MOTION PLANNING VIA OPTIMAL CONTROL FOR STOCHASTIC PROCESSES

PEYMAN MOHAJERIN ESFAHANI, DEBASISH CHATTERJEE, AND JOHN LYGEROS

ABSTRACT. We study stochastic motion planning problems which involve a controlled process, with possibly discontinuous sample paths, visiting certain subsets of the state-space while avoiding others in a sequential fashion. For this purpose, we first introduce two basic notions of motion planning, and then establish a connection to a class of stochastic optimal control problems concerned with sequential stopping-times. A weak dynamic programming principle (DPP) is then proposed, which characterizes the set of initial states that admit the existence of a policy enabling the process to execute the desired maneuver with probability at least as much as some pre-specified value. The proposed DPP consists of some auxiliary value functions defined in terms of discontinuous payoff functions. An application of the DPP is demonstrated in the context of controlled diffusion processes thereafter. It turns out that the aforementioned set of initial states can be characterized as the level set of a discontinuous viscosity solution to a sequence of partial differential equations, for which the first one has a known boundary condition, while the boundary conditions of the subsequent ones are determined by the solutions to the preceding steps. Finally, the generality and flexibility of the theoretical results are illustrated with the aid of an example involving biological switches.

1. INTRODUCTION

Motion planning can be viewed as a scheme of excursions to visiting certain specific sets in a specific order according to a specified time schedule. In the context of motion planning for controlled dynamical systems, the central issue is to determine whether there exits an admissible policy to drive the process through some sets while visiting certain targets in a pre-assigned order and scheduled times. In the deterministic setting, motion planning problems have been studied extensively from different perspectives; here we cite two representative articles [Sus91, CS98] and refer to the references therein for further details of the literature. In this article we focus on the stochastic counterpart of the motion planning. The basic motion planning problem involving two targets and obstacle sets has been investigated in different contexts, e.g., from computational standpoint in finite probability spaces for a class of continuous-time Markov decision processes (CTMDPs) [BHKH05], or in discrete-time stochastic hybrid systems based on a dynamic programming approach in [CCL11, SL10]. In our earlier works [MECL11] and [MECL12] we focused on reachability of controlled diffusion processes; the reachable sets in these works were characterized as superlevel sets of value functions which are given by the discontinuous viscosity solutions to certain partial differential equations with some associated boundary conditions. Here we continue our study by moving beyond reachability to more complex motion planning specifications for a larger class of stochastic processes with possibly discontinuous sample paths.

We first introduce different motion planning scenarios in the context of piecewise continuous processes. We address the following natural question: for which set of initial states does there

---

exist an admissible policy such that the controlled stochastic processes satisfy the motion planning specifications with a probability greater than a given value p? To characterize this set of initial states, we establish a connection between the motion planning specifications to a class of stochastic optimal control problems involved sequential stopping times. We shall be concerned with a stochastic process started and stopped when it hits certain subsets of the state-space, or equivalently, the process obtained by concatenating segments of the original process between consecutive stopping times.

Under certain mild assumptions on the admissible polices, the stochastic process, and the sets concerned the motion planning, we propose a Dynamic Programming Principle (DPP) in which some auxiliary value functions are required. The DPP is introduced in a weak version in the spirit of [BT11]; this formulation does not require measurability of the value functions. In the following sections we shall focus on a class of diffusion processes as the strong solution of a stochastic differential equation (SDE), in which the required assumptions of the DPP are investigated. In light of the proposed DPP, we develop a new framework to characterize the desired initial sets based on tools from partial differential equations (PDE's). Due to the discontinuities in the value functions corresponding to these problems, all the PDE's are understood in the generalized notion of the so called viscosity solutions. It turns out to solve the value functions a series of PDEs are considered, in which the preceding PDE provides the boundary condition of the proceeding PDE, i.e., the PDEs are solved in a recursive fashion. In order to numerically compute the desired initial sets by means of off-the-shelf PDE solvers, some numerical issues are discussed thereafter.

Mention may be made of the fact that the techniques proposed here suffer from the curse of dimensionality. Given a continuous-time controlled stochastic process, e.g., a controlled diffusion, one approach to solving motion planning problems may proceed with a discretization of the state-space, constructing a controlled Markov chain on the discretized space that approximates the original process in a certain way, and then to deal with this Markov chain insofar as motion planning is concerned. While it may appear that this discretized setting is conceptually simpler, it does not lead to dimensionality reduction, and moreover, introduces the non-trivial issue of the quality of the approximation and the associated errors due to the discretization involved. Furthermore, to our knowledge, there appears to be no off-the-shelf software that algorithmically leads to the aforementioned discretization. In contrast, our techniques deal directly with the given controlled stochastic process. The implementation of our techniques require employment of off-the-shelf PDE solvers, which is a well-studied topic. The errors in our numerical solutions occur only due to the employment of PDE solvers.

The article is organized as follows: in §2 we formally introduce the stochastic motion planning problems on the prescribed probability space. In §3 we construct a connection between the motion planning problems to a class of stochastic optimal control problems, for which a weak DPP is proposed in §4 in terms of some auxiliary value functions. An application of the proposed DPP is illustrated in §5, which leads to an alternative characterization of the motion planning objective by a series of PDE's in a recursive fashion. To validate the performance of the proposed methodology, in §6 the theoretical results are applied to a biological two-gene network, where the quality of the biological switch is investigated. For better readability, some of the technical proofs of §3 and §5 are moved to Appendix A and B, respectively.

NOTATION

For the ease of readers, we provide here a partial notation list which will be also explained in more details later throughout the article:

- $\wedge$ (resp. $\vee$): minimum (resp. maximum) operator;
- A^c (resp. A°): complement (resp. interior) of the set A;
- $\mathrm{B}_r(x)$: open Euclidian ball centered at x and radius r;
- $\mathfrak{B}(\mathbb{A})$: Borel σ-algebra on a topological space $\mathbb{A}$;
- $\mathcal{U}_t$: set of admissible policies at time t;
- $(X_s^{t,x;\boldsymbol{u}})_{s \geq 0}$: stochastic process under the control policy $\boldsymbol{u}$ and convention $X_s^{t,x;\boldsymbol{u}} := x$ for all $s \leq t$;
- $(W_i \rightsquigarrow G_i)_{\leq T_i}$ (resp. $W_i \xrightarrow{T_i} G_i$) : motion-planning events of reaching G_i sometime before time T_i (resp. at time T_i) while staying in W_i, see Definition 2.1;
- $\left(\Theta_i^{A_{k:n}}\right)_{i=k}^n$: sequential exit-times from the sets $(A_i)_{i=k}^n$ in order, see Definition 3.1;
- V^* (resp. V_*): upper (resp. lower) semicontinuous envelope of the function V;
- $\mathcal{L}^u$: Dynkin operator, see Definition 5.5.

2. General Setting and Problem Description

Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ whose filtration $\mathbb{F} := (\mathcal{F}_s)_{s \geq 0}$ is generated by an $\mathbb{R}^{d_z}$-valued process $\boldsymbol{z} := (z_s)_{s \geq 0}$ with independent increments. Let this natural filtration be enlarged by its right-continuous completion, i.e. it satisfies the usual conditions of completeness and right continuity [KS91, p. 48]. For future purposes, we introduce an auxiliary subfiltration $\mathbb{F}_t := (\mathcal{F}_{t,s})_{s \geq 0}$, where $\mathcal{F}_{t,s}$ is the $\mathbb{P}$-completion of $\sigma(z_r - z_t, t \leq r \leq t \vee s)$. Note that for $s \leq t$, $\mathcal{F}_{t,s}$ is the trivial σ-algebra, and any $\mathcal{F}_{t,s}$-random variable is independent of $\mathcal{F}_t$. By definitions, it is obvious that $\mathcal{F}_{t,s} \subseteq \mathcal{F}_s$ with equality in case of $t = 0$.

The object of our study is an $\mathbb{R}^d$-valued controlled random process $\left(X_s^{t,x;\boldsymbol{u}}\right)_{s \geq t}$, initialized at (t, x) under the control policy $\boldsymbol{u} \in \mathcal{U}_t$, where $\mathcal{U}_t$ is the set of admissible policies at time t. Let $T > 0$ be a fixed time horizon, and let $\mathbb{S} := [0, T] \times \mathbb{R}^d$. Throughout this work we assume that for every $(t, x) \in \mathbb{S}$ and $\boldsymbol{u} \in \mathcal{U}_t$, the process $\left(X_s^{t,x;\boldsymbol{u}}\right)_{s \geq t}$ is $\mathbb{F}$-adapted process with RCLL sample paths.[1] We denote by $\mathcal{T}$ the collection of all $\mathbb{F}$-stopping times; for $\tau_1, \tau_2 \in \mathcal{T}$ with $\tau_1 \leq \tau_2$ $\mathbb{P}$-a.s. we let the subset $\mathcal{T}_{[\tau_1, \tau_2]}$ denote the collection of all $\mathbb{F}_{\tau_1}$-stopping times τ such that $\tau_1 \leq \tau \leq \tau_2$ $\mathbb{P}$-a.s. Measurability on $\mathbb{R}^d$ will always refer to Borel-measurability, and $\mathfrak{B}(\mathbb{A})$ stands for the Borel σ-algebra on a topological space $\mathbb{A}$. Throughout this article all the (in)equalities between random variables are understood in almost sure sense.

Given sets $(W_i, G_i) \in \mathfrak{B}(\mathbb{R}^d) \times \mathfrak{B}(\mathbb{R}^d)$ for $i \in \{1, \cdots, n\}$, we are interested in a set of initial conditions $(t, x) \in \mathbb{S}$ such that there exists an admissible strategy $\boldsymbol{u} \in \mathcal{U}_t$ steering the process $X^{t,x;\boldsymbol{u}}$ through the sets $(W_i)_{i=1}^n$ while visiting $(G_i)_{i=1}^n$ in a pre-assigned order. In fact, W_i and G_i stand for *"Way"* and *"Goal"* respectively. One may pose this objective from different perspectives based on different time scheduling for the excursions between the sets. We formally introduce some of these notions which will be addressed throughout this article.

Definition 2.1 (Motion-Planning Events). *Consider a fixed initial condition $(t, x) \in \mathbb{S}$ and admissible policy $\boldsymbol{u} \in \mathcal{U}_t$. Given a sequence of pairs $(W_i, G_i)_{i=1}^n \subset \mathfrak{B}(\mathbb{R}^d) \times \mathfrak{B}(\mathbb{R}^d)$ and horizon times $(T_i)_{i=1}^n \subset [t, T]$, we introduce the following **motion-planning events**:*

(1a) $\left\{X_\cdot^{t,x;\boldsymbol{u}} \models \left[(W_1 \rightsquigarrow G_1) \circ \cdots \circ (W_n \rightsquigarrow G_n)\right]_{\leq T}\right\} :=$

$$\left\{\exists (s_i)_{i=1}^n \subset [t, T] \mid X_{s_i}^{t,x;\boldsymbol{u}} \in G_i \text{ and } X_r^{t,x;\boldsymbol{u}} \in W_i \setminus G_i, \ \forall r \in [s_{i-1}, s_i[, \quad \forall i \leq n\right\},$$

(1b) $\left\{X_\cdot^{t,x;\boldsymbol{u}} \models (W_1 \xrightarrow{T_1} G_1) \circ \cdots \circ (W_n \xrightarrow{T_n} G_n)\right\} :=$

---

[1]That is, processes with paths that are right continuous with left limits.

$$\left\{ X_{T_i}^{t,x;\boldsymbol{u}} \in G_i \ \text{and} \ X_r^{t,x;\boldsymbol{u}} \in W_i, \quad \forall r \in [T_{i-1}, T_i], \quad \forall i \leq n \right\},$$

where in the above definitions $s_0 = T_0 := t$.

The set in (1a), roughly speaking, contains those events that the trajectory $X_{\cdot}^{t,x;\boldsymbol{u}}$, initialized at $(t,x) \in \mathbb{S}$ and controlled via $\boldsymbol{u} \in \mathcal{U}_t$, succeeds in visiting the sets $(G_i)_{i=1}^n$ in a certain order, while the entire duration between the two visits to G_{i-1} and G_i is spent in W_i, all within the time horizon T. In other words, the journey from G_{i-1} to the next destination G_i must belong to the way W_i for all i. Figure 1(a) depicts a sample path that successfully contributes to the first three phases of the excursion in the sense of (1a).
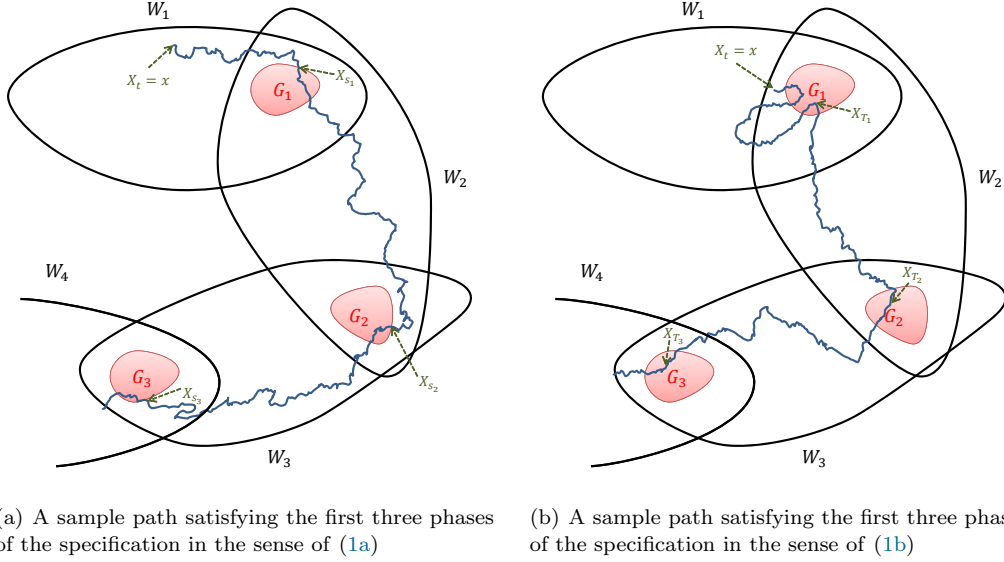

(a) A sample path satisfying the first three phases of the specification in the sense of (1a)

(b) A sample path satisfying the first three phases of the specification in the sense of (1b)

FIGURE 1. Sample paths of the process $X_{\cdot}^{t,x;\boldsymbol{u}}$ for a fix policy $\boldsymbol{u} \in \mathcal{U}_t$

In the case of (1b), the set of paths is more restricted in comparison to (1a). Indeed, not only is the trajectory confined to the ways W_i, but also there is a time schedule $(T_i)_{i=1}^n$ that a priori forces the process to be at the goal sets G_i at the specific times $(T_i)_{i=1}^n$. Figure 1(b) demonstrates one sample path in which the first three phases of the excursion are successfully fulfilled.

Note that once a trajectory belonging ot the set in (1a) visits G_i for the first time, it is required to remain in the way W_{i+1} until the next goal G_{i+1} is reached, whereas a trajectory belonging to the set in definition (1b) may visit the destination G_i several times, while staying in W_i, until the intermediate time schedule T_i. The only requirement, in contrast to (1a), is to confine the trajectory to be at the goal G_i at the time T_i.

As an illustration, one can easily inspect that the successful sample path in Figure 1(b) indeed violates the requirements of the definition (1a) as it leaves W_2 after it visits the goal set G_1 for the first time. In other words, the definition (1a) changes the admissible way set W_i to W_{i+1} *immediately after* the trajectory visits the goal set G_i, while the definition (1b) only changes the admissible way set *only after* the intermediate time T_i irrespective of whether the trajectory visits the goal set G_i prior to T_i.

From the technical standpoint, if the target set G_i is not closed, then it is not difficult to see that there could be some continuous transitions through the boundary of the goal G_i that are not admissible in view of the definition (1a) since the trajectory must reside in $W_i \setminus G_i$ for the whole interval $[s_{i-1}, s_i[$ and just hit the set G_i at the time s_i. Notice that we do not need to consider this issue for the set in definition (1b) since in this case the trajectory only visits the sets G_i at the specific times T_i while any continuous transition and maneuver inside the target sets G_i are allowed. In order to address the aforementioned issue, we may impose the following:

Assumption 2.2. *The sets $(G_i)_{i=1}^n \subset \mathfrak{B}(\mathbb{R}^d)$ are closed.*

Having discussed about the properties of motion planning as detailed in Definition 2.1, one may conclude that in general none of the definitions of (1) is more restrictive than the other. However, in the particular motion planning case where the family of the way sets $(W_i)_{i=1}^n$ is nested, i.e., $W_i \subseteq W_{i+1}$, one can see that the motion planning definition of (1b) actually imposes more constraints on the trajectories of the random process in light of the right continuity of $X_{\cdot}^{t,x;\boldsymbol{u}}$ and Assumption 2.2. The following Fact formally addresses this issue.

Fact 2.3. *Consider a family of set pairs $(W_i, G_i)_{i=1}^n \subset \mathfrak{B}(\mathbb{R}^d) \times \mathfrak{B}(\mathbb{R}^d)$ where $(G_i)_{i=1}^n$ satisfies Assumption 2.2 and $(W_i)_{i=1}^n$ is nested, i.e. $W_i \subseteq W_{i+1}$. Then, for all initial condition $(t, x) \in \mathbb{S}$, intermediate times $(T_i)_{i=1}^n \subset [t, T]$, and control policy $\boldsymbol{u} \in \mathcal{U}_t$, it holds that*

$$\left\{ X_{\cdot}^{t,x;\boldsymbol{u}} \models (W_1 \xrightarrow{T_1} G_1) \circ \cdots \circ (W_n \xrightarrow{T_n} G_n) \right\} \subseteq \left\{ X_{\cdot}^{t,x;\boldsymbol{u}} \models \left[(W_1 \rightsquigarrow G_1) \circ \cdots \circ (W_n \rightsquigarrow G_n) \right]_{\leq T} \right\}$$

Proof. Let ω be contained in the set defined in (1b). This means that $X_{T_i}^{t,x;\boldsymbol{u}}(\omega) \in G_i$, and for all $r \in [T_{i-1}, T_i]$ we have $X_r^{t,x;\boldsymbol{u}}(\omega) \in W_i$. Due to the right continuity of the sample paths and Assumption 2.2, one can see that for all $i \in \{1, \cdots, n\}$ there exists $s_i \in [T_{i-1}, T_i]$ so that $X_{s_i}^{t,x;\boldsymbol{u}}(\omega) \in G_i$ while $X_r^{t,x;\boldsymbol{u}}(\omega) \in W_i \setminus G_i$ for all $r \in [T_{i-1}, s_i[$, where $T_0 := t$. Since the sets $(W_i)_{i=1}^n$ are nested, $X_r^{t,x;\boldsymbol{u}}(\omega) \in W_{i-1} \subseteq W_i$ for all $r \in [s_{i-1}, T_{i-1}]$. An induction argument quickly leads to

$$X_{s_i}^{t,x;\boldsymbol{u}}(\omega) \in G_i,$$
$$X_r^{t,x;\boldsymbol{u}}(\omega) \in W_i \setminus G_i, \quad \forall r \in [s_{i-1}, s_i[,$$

where $s_0 := t$. In fact one may introduce s_i as the first hitting time of the set G_i after time T_{i-1}.[2] This implies that ω is also contained in the set defined in (1a) and proves the assertion. □

Let us note that to satisfy the order of visiting the goal/target sets $(G_i)_{i=1}^n$, it only suffices to exclude the target sets $(G_i)_{i=k+1}^n$ from the set W_k. Therefore, the hypothesis in Fact 2.3 concerning the nested way sets W_i does not really impose any restriction on the motion planning objectives.

Remark 2.4. The motion planning scenarios for only two sets (W_1, G_1) are essentially the basic reachability maneuver that was studied in our earlier work [MECL12]. The definition (1a) suggests the same Reach-Avoid problem as in [MECL12, Definition 2.4], where the target and obstacle sets are G_1 and $\mathbb{R}^d \setminus W_1$ respectively. In this special case, the definition (1b) also follows the same concept as Reach-Avoid problem in [MECL12, Definition 3.4] with the same target and obstacle sets.

Remark 2.5. A particular case of the Definition 2.1 is the following: $G_i := A_{i+1} \setminus A_i$ and $G_i := A_{i+1} \cap A_i$ for the definitions (1a) and (1b) respectively. Here the motion planning objective is to pass through n given sets $(A_i)_{i=1}^n$ in a certain order.

---

[2]The formal definition of this stopping time and its application are considered later for Fact 3.3.

The events introduced in Definition 2.1 depend, of course, on the control policy $\boldsymbol{u} \in \mathcal{U}$ and initial condition $(t, x) \in \mathbb{S}$. The central objective of this work is to determine the set of initial conditions $x \in \mathbb{R}^d$ such that there exists an admissible policy $\boldsymbol{u}$ where the probability of the above path-planning events is higher than a certain threshold. To this end, we formally introduce these sets as follows:

Definition 2.6 (Motion-Planning Initial Set). *Consider a fixed initial time $t \in [0, T]$. Given a sequence of set pairs $(W_i, G_i)_{i=1}^n \subset \mathfrak{B}(\mathbb{R}^d) \times \mathfrak{B}(\mathbb{R}^d)$ and horizon times $(T_i)_{i=1}^n \subset [t, T]$, we define the following* **motion-planning initial sets**:

(2a) $\mathrm{PP}\big(t, p;\, (W_i, G_i)_{i=1}^n, T\big) :=$
$$\Big\{ x \in \mathbb{R}^d \mid \exists \boldsymbol{u} \in \mathcal{U}_t \; : \; \mathbb{P}\big\{ X_{\cdot}^{t,x;\boldsymbol{u}} \models \big[(W_1 \rightsquigarrow G_1) \circ \cdots \circ (W_n \rightsquigarrow G_n)\big]_{\leq T} \big\} > p \Big\},$$

(2b) $\widetilde{\mathrm{PP}}\big(t, p;\, (W_i, G_i)_{i=1}^n, (T_i)_{i=1}^n\big) :=$
$$\Big\{ x \in \mathbb{R}^d \mid \exists \boldsymbol{u} \in \mathcal{U}_t \; : \; \mathbb{P}\big\{ X_{\cdot}^{t,x;\boldsymbol{u}} \models (W_1 \xrightarrow{T_1} G_1) \circ \cdots \circ (W_n \xrightarrow{T_n} G_n) \big\} > p \Big\}.$$

3. Connection to Stochastic Optimal Control

In this section we establish a connection from stochastic motion-planning initial sets PP and $\widetilde{\mathrm{PP}}$, defined in Definition 2.6, and a class of stochastic optimal control problems involving stopping times. We introduce a sequence of random times that corresponds to the times that the process $X_{\cdot}^{t,x;\boldsymbol{u}}$ for the first time exits from the sequence of sets one after another in a certain order:

Definition 3.1. *Given an initial condition $(t, x) \in \mathbb{S}$ and a sequence of measurable sets $(A_i)_{i=k}^n \subset \mathfrak{B}(\mathbb{R}^d)$, the sequence of random times $\big(\Theta_i^{A_{k:n}}\big)_{i=k}^n$ defined[3] by*
$$\Theta_i^{A_{k:n}}(t, x) := \inf \big\{ r \geq \Theta_{i-1}^{A_{k:n}}(t, x) \; : \; X_r^{t,x;\boldsymbol{u}} \notin A_i \big\}, \qquad \Theta_{k-1}^{A_{k:n}}(t, x) := t,$$
is called the **sequential exit-time** *through the set A_k to A_n.*

Note that the sequential exit-time $\Theta_i^{A_{k:n}}$ depends on the control policy $\boldsymbol{u}$ in addition to the initial condition (t, x), but here and later in the sequel we shall suppress this dependence. For notational simplicity, we also drop (t, x) in the subsequent sections.

In Figure 2 a sample path of the process $X_{\cdot}^{t,x;\boldsymbol{u}}$ along with the sequential exit-times $(\Theta_i^{A_{k:3}})_{i=k}^n$ is depicted for different $k \in \{1, \cdots, 3\}$. Note that since the initial condition x does not belongs to A_3, the first exit-time of the set A_3 is indeed the start time t, i.e., $\Theta_3^{A_{3:3}} = t$. Let us highlight the difference between stopping times $\Theta_2^{A_{1:3}}$ and $\Theta_2^{A_{2:3}}$. The former is the first exit-time of the set A_2 after the time that the process leaves A_1, whereas the latter is the first exit-time of the set A_2 from the very beginning.[4]

Given a stopping time $\theta \in \mathcal{T}_{[t,T]}$, let $\omega \in \Omega$ be a sample realization such that the process $\big(X_s^{t,x;\boldsymbol{u}}(\omega)\big)_{t \leq s \leq \theta(\omega)}$ successfully visits the first sets $(A_j)_{j=1}^{i-1}$. For this ω, it then follows from Definition 3.1 that the sequential stopping times $\Theta_j^{A_{1:n}}$ of visiting the sets $(A_j)_{j=i}^n$ starting at the initial condition (t, x) is the same as the sequential exit-times $\Theta_j^{A_{i:n}}$ of visiting the sets $(A_j)_{j=i}^n$ starting at $(\theta(\omega), X_{\theta(\omega)}^{t,x;\boldsymbol{u}})$. Figure 3 depicts one of these sample paths where $\Theta_1^{A_{1:3}}(\omega) \leq \theta(\omega) < \Theta_1^{A_{1:3}}(\omega)$. It is obvious from the figure that starting from an initial condition (t, x),

---

[3]By convention, $\inf \emptyset = \infty$.

[4]In §5 we shall see that these differences will lead to different definitions of value functions in order to derive a dynamic programming argument.
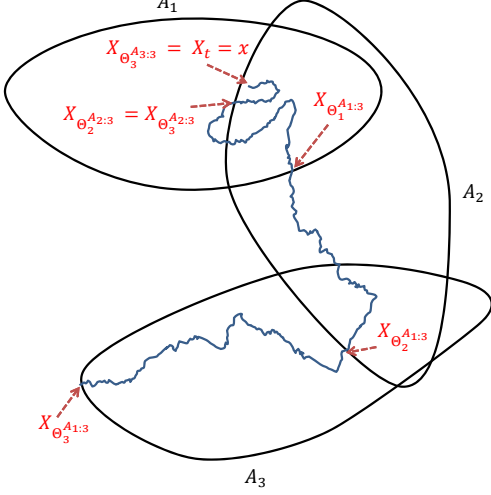
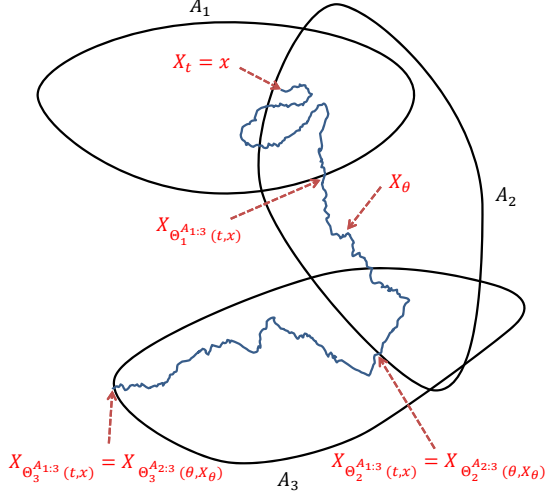FIGURE 2. Sequential exit-times of a sample path through the sets $(A_i)_{i=k}^3$ for different values of k

FIGURE 3. Sequential exit-times corresponding to different initial conditions

the process views the exit-times of A_2 and A_3 after leaving A_1 as if it starts from the initial condition $(\theta, X_\theta^{t,x;\boldsymbol{u}})$ and proceeds to exit from the set A_2 and A_3 irrespective of the history before θ. Lemma 3.2 formally presents this result and will be employed later to prove one of the main result of this article, i.e., Theorem 4.5.

Lemma 3.2. *Consider a control policy $\boldsymbol{u} \in \mathcal{U}_t$ and initial condition $(t, x) \in \mathbb{S}$. Given a sequence of measurable sets $(A_i)_{i=k}^n \subset \mathfrak{B}(\mathbb{R}^d)$ and stopping time $\theta \in \mathcal{T}_{[t,T]}$, for all $k \in \{1, \cdots, n\}$ and $j \geq i \geq k$ we have*

$$\Theta_j^{A_{k:n}}(t, x) = \Theta_j^{A_{i:n}}(\theta, X_\theta^{t,x;\boldsymbol{u}}) \qquad on \quad \{\Theta_{i-1}^{A_{k:n}}(t, x) \leq \theta < \Theta_i^{A_{k:n}}(t, x)\}$$

We need to establish that the sequential stopping-times are well-defined, hence:

Fact 3.3 (Measurability). *Consider a sequence of measurable sets $(A_i)_{i=1}^n \subset \mathfrak{B}(\mathbb{R}^d)$ and initial condition $(t, x) \in \mathbb{S}$. The sequential exit-time $\Theta_i^{A_{1:n}}(t, x)$ is an $\mathbb{F}_t$-stopping time for all $i \in \{1, \cdots, n\}$, i.e., $\{\Theta_i^{A_{1:n}}(t, x) \leq s\} \in \mathcal{F}_{t,s}$ for all $s \geq 0$.*

Proof. Let τ_A be the first exit-time from the set A_i:

$$(3) \qquad \tau_{A_i}(t, x) := \inf\{s \geq 0 \ : \ X_{t+s}^{t,x;\boldsymbol{u}} \notin A_i\}.$$

We know that τ_A is an $\mathbb{F}_t$-stopping time [EK86, Theorem 1.6, Chapter 2]. Let $\omega(\cdot) \mapsto \vartheta_s(\omega(\cdot)) := \omega(s + \cdot)$ be the time-shift operator. From the definition it follows that for all $i \geq 0$

$$\Theta_{i+1}^{A_{1:n}} = \Theta_i^{A_{1:n}} + \tau_{A_i} \circ \vartheta_{\Theta_i^{A_{1:n}}}.$$

Now the assertion follows directly in light of the measurability of the mapping ϑ and right continuity of the filtration $\mathbb{F}_t$.[5] $\qquad\square$

For technical reasons we stipulate that the way sets satisfy the following:

---

[5]See, for instance, [EK86, Proposition 1.4, Chapter 2] for more details in this regard.

Assumption 3.4. *The sets $(W_i)_{i=1}^n \subset \mathfrak{B}(\mathbb{R}^d)$ are open.*

Under Assumption 3.4, the following fact is an immediate consequence of right continuity of the stochastic process $X_\cdot^{t,x;\boldsymbol{u}}$:

Fact 3.5. *Fix a control policy $\boldsymbol{u} \in \mathcal{U}_t$ and an initial condition $(t,x) \in \mathbb{S}$. Let measurable sets $(W_i)_{i=1}^n \subset \mathfrak{B}(\mathbb{R}^d)$ be given, and suppose that Assumption 3.4 holds for $(W_i)_{i=1}^n$. Then, for all $i \in \{1, \cdots, n\}$*

$$X_{\Theta_i^{W_{1:n}}}^{t,x;\boldsymbol{u}} \notin W_i, \qquad on \quad \{\Theta_i^{W_{1:n}} < \infty\},$$

where $\left(\Theta_i^{W_{1:n}}\right)_{i=1}^n$ are the sequential exit-times introduced as in Definition 3.1.

Given $(W_i, G_i, T_i)_{i=1}^n \subset \mathfrak{B}(\mathbb{R}^d) \times \mathfrak{B}(\mathbb{R}^d) \times [t,T]$, we introduce two value functions $V, \widetilde{V} : \mathbb{S} \to [0,1]$ defined by

$$(4a) \quad V(t,x) := \sup_{\boldsymbol{u} \in \mathcal{U}_t} \mathbb{E}\left[\prod_{i=1}^n \mathbb{1}_{G_i}\left(X_{\eta_i}^{t,x;\boldsymbol{u}}\right)\right], \qquad \eta_i := \Theta_i^{B_{1:n}} \wedge T, \qquad B_i := W_i \setminus G_i,$$

$$(4b) \quad \widetilde{V}(t,x) := \sup_{\boldsymbol{u} \in \mathcal{U}_t} \mathbb{E}\left[\prod_{i=1}^n \mathbb{1}_{G_i \cap W_i}\left(X_{\widetilde{\eta}_i}^{t,x;\boldsymbol{u}}\right)\right], \qquad \widetilde{\eta}_i := \Theta_i^{W_{1:n}} \wedge T_i,$$

where $\Theta_i^{W_{1:n}}, \Theta_i^{B_{1:n}}$ are the sequential exit-times in the sense of Definition 3.1. Figure 4(a) and 4(b) illustrate the sequential exit-times corresponding to the sets B_i and W_i, respectively; the sample trajectories are the same as Figure 1. The main result of this section, Proposition 3.6 below, establishes a connection from the sets $\mathrm{PP}, \widetilde{\mathrm{PP}}$ and superlevel sets of the value functions V and $\widetilde{V}$.

(a) Sequential exit-times related to motion-planning event (1a)

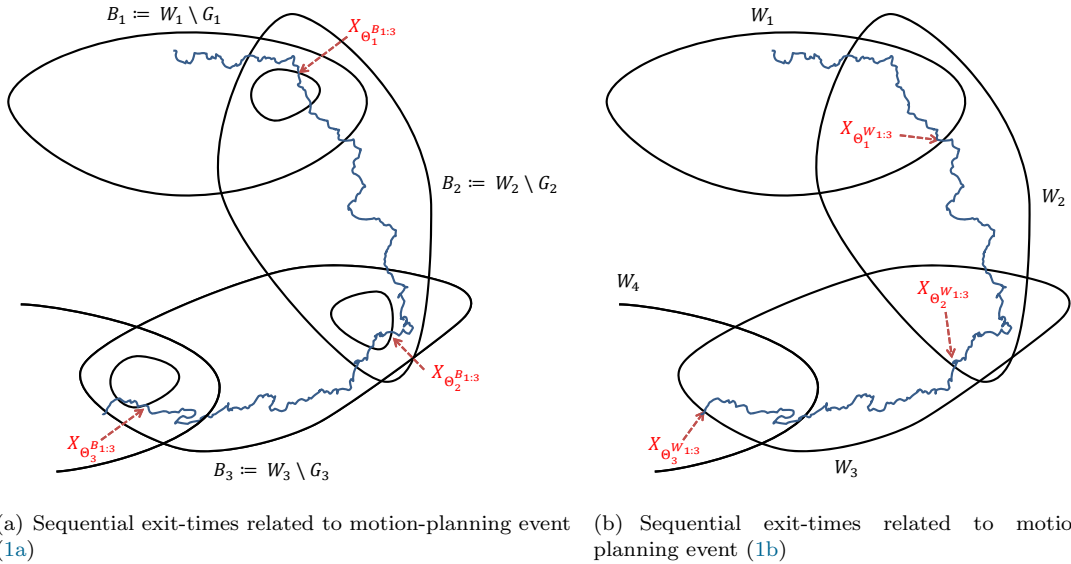(b) Sequential exit-times related to motion-planning event (1b)

FIGURE 4. Sequential exit-times corresponding to different motion-planning events as introduced in (1)

Proposition 3.6. *Fix a probability level $p \in [0, 1]$, a sequence of set pairs $(W_i, G_i)_{i=1}^n \subset \mathfrak{B}(\mathbb{R}^d) \times \mathfrak{B}(\mathbb{R}^d)$, an initial time $t \in [0, T]$, and intermediate times $(T_i)_{i=1}^n \subset [t, T]$. Then*

$$(5) \qquad \mathrm{PP}\big(t, p; (W_i, G_i)_{i=0}^n, T\big) = \big\{ x \in \mathbb{R}^d \mid V(t, x) > p \big\}.$$

Moreover, suppose Assumption 3.4 holds. Then,

$$(6) \qquad \widetilde{\mathrm{PP}}\big(t, p; (W_i, G_i)_{i=0}^n, (T_i)_{i=1}^n\big) = \big\{ x \in \mathbb{R}^d \mid \widetilde{V}(t, x) > p \big\},$$

where the value functions V and $\widetilde{V}$ are as defined in (4).

Proof. See Appendix A. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Remark 3.7 (Mixed Motion-Planning Events). In this section we focus on two sets of events as introduced in Definition 2.1, and will continue doing so for our subsequent results. However, it is of interest to consider an event that consists of a mixture of the events in (1), e.g., $(W_1 \rightsquigarrow G_1)_{\leq T_1} \circ (W_2 \xrightarrow{T_2} G_2)$. One can observe that essentially the same analytical techniques as the ones proposed here can be employed to address these mixed motion planning objectives, and establish a connection to a class of optimal control problems with some appropriate sequential stopping times. We shall provide an example of this nature in §6.

4. Dynamic Programming Principle

The objective of this section is to derive a DPP for the value functions V and $\widetilde{V}$ introduced in (4). We proceed with a more abstract setting for transparent elucidation in the following fashion: Let $(T_i)_{i=1}^n \subset [0, T]$ be a sequence of times, $(A_i)_{i=1}^n \subset \mathfrak{B}(\mathbb{R}^n)$ be a family of open sets, and payoff functions $\ell_i : \mathbb{R}^r \to \mathbb{R}$ that are measurable and bounded, $i = 1, \ldots, n$. We define the sequence of value functions $V_k : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $k = 1, \ldots, n$,

$$(7) \qquad V_k(t, x) := \sup_{\boldsymbol{u} \in \mathcal{U}_t} \mathbb{E}\Big[\prod_{i=k}^n \ell_i\big(X_{\tau_i^k}^{t,x;\boldsymbol{u}}\big) \Big], \qquad \tau_i^k(t, x) := \Theta_i^{A_{k:n}}(t, x) \wedge T_i, \quad i \in \{k, \cdots, n\},$$

where the stopping times $(\Theta_i^{W_{k:n}})_{i=k}^n$ are sequential exit-times in the sense of Definition 3.1. Notice that the sequential exit-times of the value function V_k correspond to an excursion through the sets $(A_i)_{i=k}^n$ irrespective of the first $(k-1)$ sets. It is straightforward to observe that the value functions V and $\widetilde{V}$ in (4) are particular cases of the value function V_1 defined as in (7) for an appropriate selection of the sets $(A_i)_{i=1}^n$, functions $(\ell_i)_{i=1}^n$, and intermediate times $(T_i)_{i=1}^n$.

To state the main result of this section, Theorem 4.5 below, some technical definitions and assumptions concerning the stochastic processes $X_{\cdot}^{t,x;\boldsymbol{u}}$, admissible strategies $\mathcal{U}_t$, and the payoff functions ℓ_i, are needed: Let $\mathbb{A}$ be a metric space, and let $f : \mathbb{A} \longrightarrow \mathbb{R}$ be a function. The lower and upper semicontinuous envelopes, respectively, of f are defined as:

$$f_*(x) := \liminf_{x' \to x} f(x') \qquad f^*(x) := \limsup_{x' \to x} f(x').$$

We denote by $\mathrm{USC}(\mathbb{A})$ and $\mathrm{LSC}(\mathbb{A})$ the collection of all upper-semicontinuous and lower-semicontinuous functions from $\mathbb{A}$ to $\mathbb{R}$, respectively.

Assumption 4.1. *We stipulate the following assumptions on*

a. ***Admissible control policies:***
    i. Non-anticipative policy: *Given a control set $\mathbb{U} \subset \mathbb{R}^{d_u}$, any $\boldsymbol{u} \in \mathcal{U}_t$ is $\mathbb{U}$-valued $\mathbb{F}_t$-adapted stochastic process;*

   ii. Stability under concatenation: *The set of admissible control policies at time t, $\mathcal{U}_t$, is closed under concatenation. That is, for all $\boldsymbol{u_1}, \boldsymbol{u_2} \in \mathcal{U}_t$ and stopping time $\theta \in \mathcal{T}_{[t,T]}$, it holds that*

$$\mathbb{1}_{[t,\theta]}\boldsymbol{u_1} + \mathbb{1}_{]\theta,T]}\boldsymbol{u_2} \in \mathcal{U}_t;$$

b. **Stochastic process $X_.^{t,x;\boldsymbol{u}}$:**

   i. Causality: *For all initial conditions $(t,x) \in \mathbb{S}$, any control policies $\boldsymbol{u_1}, \boldsymbol{u_2} \in \mathcal{U}_t$, and stopping time $\theta \in \mathcal{T}_{[t,T]}$, it holds that*

$$\mathbb{1}_{[t,\theta]}\boldsymbol{u_1} = \mathbb{1}_{[t,\theta]}\boldsymbol{u_2} \quad \implies \quad \mathbb{1}_{[t,\theta]}X_.^{t,x;\boldsymbol{u_1}} = \mathbb{1}_{[t,\theta]}X_.^{t,x;\boldsymbol{u_2}}$$

   ii. Strong Markov property: *Given initial condition $(t,x) \in \mathbb{S}$, control policy $\boldsymbol{u} \in \mathcal{U}_t$, and stopping time $\theta \in \mathcal{T}_{[t,T]}$, for all bounded measurable functions $\ell : \mathbb{R}^n \to \mathbb{R}$ and $s \geq 0$ it holds that*

$$\mathbb{E}\Big[\ell\big(X_{\theta+s}^{t,x;\boldsymbol{u}}\big)\Big|\mathcal{F}_\theta\Big] = \mathbb{E}\Big[\ell\big(X_{\theta+s}^{t,x;\boldsymbol{u}}\big)\Big|X_\theta^{t,x;\boldsymbol{u}}\Big] \qquad \mathbb{P}\text{-}a.s.;$$

   iii. Continuity of exit-time: *Given initial condition $(t_0, x_0) \in \mathbb{S}$ and control policy $\boldsymbol{u} \in \mathcal{U}_t$, for all $k \in \{1, \cdots, n\}$ and $i \in \{k, \cdots, n\}$ the stochastic mapping $(t,x) \mapsto X_{\tau_i^k(t,x)}^{t,x;\boldsymbol{u}}$ is $\mathbb{P}$-a.s. continuous at the point (t_0, x_0) where the stopping time τ_i^k is defined as in* (7);

c. **Payoff functions:**

   $(\ell_i)_{i=1}^n$ *are lower semicontinuous, i.e., $\ell_i \in \mathrm{LSC}(\mathbb{R}^d)$ for all $i \leq n$.*

Remark 4.2. Some remarks on the above assumptions are in order:

○ Assumption 4.1.a.i. implies that admissible strategies $\boldsymbol{u} \in \mathcal{U}_t$ take action at time t independent of future information arriving at $s > t$. This is known as a *non-anticipative* strategy [Bor05], and is a standard assumption.

○ Assumption 4.1.a.ii. is also standard for dynamic programming arguments and it holds for a large class of admissible strategies, e.g., progressively measurable processes.

○ Assumption 4.1.b. imposes three constraints on the process $X_.^{t,x;\boldsymbol{u}}$ defined on the prescribed probability space: i) causality of the solution processes for a given admissible policy ii) strong Markov property iii) continuity of exit-time. The causality property is always satisfied in practical applications; uniqueness of the solution process $X_.^{t,x;\boldsymbol{u}}$ under any admissible control process $\boldsymbol{u}$ guarantees it. The class of Markovian processes is fairly large; for instance, it contains the solution of SDEs under some mild assumptions on the drift and diffusion terms [Kry09, Theorem 2.9.4]. The almost sure continuity of the exit-time with respect to the initial condition of the process is the only restrictive. Note that this condition does not always hold even for deterministic processes with continuous trajectories. One may need to impose conditions on the process and possibly the sets involved in motion planning in order to satisfy continuity of the mapping $(t,x) \mapsto X_{\tau_i^k(t,x)}^{t,x;\boldsymbol{u}}$ at the given initial condition with probability one. We shall elaborate on this issue and its ramifications to a class of diffusions in §5.

We present two Lemmas preparatory to our main result. For $k = 1, \ldots, n$, we define the function $J_k : \mathbb{S} \times \mathcal{U} \to \mathbb{R}$ as:

$$(8) \qquad J_k(t,x;\boldsymbol{u}) := \mathbb{E}\Big[\prod_{i=k}^n \ell_i\big(X_{\tau_i^k}^{t,x;\boldsymbol{u}}\big)\Big],$$

where $\big(\tau_i^k\big)_{i=k}^n$ are as defined in (7).

Lemma 4.3. *Consider the value function V_k and J_k as defined in* (7) *and* (8) *respectively. Then, for any $k \in \{1, \cdots, n\}$*

$$V_k(t,x) = \sup_{\boldsymbol{u} \in \mathcal{U}} J_k(t,x;\boldsymbol{u}), \qquad \forall (t,x) \in \mathbb{S}.$$

Proof. Note that all the admissible policies $\boldsymbol{u} \in \mathcal{U}_t$ are also contained in $\mathcal{U}$. Therefore, the inequality " $\geq$ " between the left- and right-hand sides is immediate. In order to show the reverse inequality, observe that any control policy $\boldsymbol{u} \in \mathcal{U}$ is a measurable function of the process $(z_s)_{s \geq 0}$. For any fixed $(z_s)_{0 \leq s \leq t}$ it holds that $\boldsymbol{u} := \boldsymbol{u}\big((z_s)_{0 \leq s \leq T}\big) = \widetilde{\boldsymbol{u}}\big((z_s)_{0 \leq s \leq t}, (z_s - z_t)_{t \geq s \geq T}\big)$, where $\widetilde{\boldsymbol{u}}$ can be viewed as a policy independent of $\mathcal{F}_t$ due to independent increments of z. By [BT11, Remark 5.2] the inequality " $\leq$ " also follows, and the assertion of the lemma follows.[6] $\square$

Lemma 4.4. *Under Assumptions 4.1.b.iii. and 4.1.c., the function $\mathbb{S} \ni (t, x) \mapsto J_k(t, x; \boldsymbol{u}) \in \mathbb{R}$ is lower semicontinuous for all $k \in \{1, \cdots, n\}$ and control policy $\boldsymbol{u} \in \mathcal{U}$.*

Proof. Fix $k \in \{1, \ldots, n\}$. It is obvious that the function J_k is uniformly bounded since ℓ_k are. Therefore,

$$\liminf_{(s,y) \to (t,x)} J_k(s, y; \boldsymbol{u}) = \liminf_{(s,y) \to (t,x)} \mathbb{E}\Big[\prod_{i=k}^{n} \ell_i\big(X_{\tau_i^k(s,y)}^{s,y;\boldsymbol{u}}\big)\Big] \geq \mathbb{E}\Big[\liminf_{(s,y) \to (t,x)} \prod_{i=k}^{n} \ell_i\big(X_{\tau_i^k(s,y)}^{s,y;\boldsymbol{u}}\big)\Big]$$

$$\geq \mathbb{E}\Big[\prod_{i=k}^{n} \liminf_{(s,y) \to (t,x)} \ell_i\big(X_{\tau_i^k(s,y)}^{s,y;\boldsymbol{u}}\big)\Big] \geq \mathbb{E}\Big[\prod_{i=k}^{n} \ell_i\big(X_{\tau_i^k(s,y)}^{t,x;\boldsymbol{u}}\big)\Big] = J_k(t, x; \boldsymbol{u}),$$

where the inequality in the first line follows from the Fatou's lemma, and the second inequality in the second line is a direct consequence of Assumptions 4.1.b.iii. and 4.1.c. $\square$

The following Theorem, the main result of this section, establishes a dynamic programming argument for the value function V_k in terms of the "successor" value functions $(V_j)_{j=k+1}^n$, all defined as in (7). For ease of notation we shall introduce deterministic times $\tau_{k-1}^k, \tau_{n+1}^k$, and a trivial constant value function V_{n+1}.

Theorem 4.5. *Consider the value functions $(V_j)_{j=1}^n$ and the sequential stopping times $(\tau_j^k)_{j=k}^n$ introduced in (7). Under Assumption 4.1, for all initial conditions $(t, x) \in \mathbb{S}$, stopping times $\theta \in \mathcal{T}_{[t,T]}$, $k \in \{1, \cdots, n\}$ we have*

$$(9a) \qquad V_k(t, x) \leq \sup_{\boldsymbol{u} \in \mathcal{U}_t} \mathbb{E}\Bigg[\sum_{j=k}^{n+1} \mathbb{1}_{\{\tau_{j-1}^k \leq \theta < \tau_j^k\}} V_j^*\big(\theta, X_\theta^{t,x;\boldsymbol{u}}\big) \prod_{i=k}^{j-1} \ell_i\big(X_{\tau_i^k}^{t,x;\boldsymbol{u}}\big)\Bigg],$$

$$(9b) \qquad V_k(t, x) \geq \sup_{\boldsymbol{u} \in \mathcal{U}_t} \mathbb{E}\Bigg[\sum_{j=k}^{n+1} \mathbb{1}_{\{\tau_{j-1}^k \leq \theta < \tau_j^k\}} V_{j*}\big(\theta, X_\theta^{t,x;\boldsymbol{u}}\big) \prod_{i=k}^{j-1} \ell_i\big(X_{\tau_i^k}^{t,x;\boldsymbol{u}}\big)\Bigg],$$

where V_j^* and V_{j*} are upper and lower semicontinuous envelopes of the value function V_j, respectively, $\tau_{k-1}^k := t$, $V_{n+1} \equiv 1$, and τ_{n+1}^k can be chosen any constant time strictly greater than T, say $\tau_{n+1}^k := T + 1$.

Proof. The proof extends the main result of our earlier work [MECL12, Theorem 4.7] on the so-called "reach-avoid" motion planning maneuver. Based on the tower property of conditional expectation [Kal97, Theorem 5.1], we have

$$\mathbb{E}\Big[\prod_{i=k}^{n} \ell_i\big(X_{\tau_i^k}^{t,x;\boldsymbol{u}}\big)\Big|\mathcal{F}_\theta\Big] = \sum_{j=k}^{n+1} \mathbb{1}_{\{\tau_{j-1}^k \leq \theta < \tau_j^k\}} \mathbb{E}\Big[\prod_{i=j}^{n} \ell_i\big(X_{\tau_i^k}^{t,x;\boldsymbol{u}}\big)\Big|\mathcal{F}_\theta\Big] \prod_{i=k}^{j-1} \ell_i\big(X_{\tau_i^k}^{t,x;\boldsymbol{u}}\big)$$

$$(10a) \qquad = \sum_{j=k}^{n+1} \mathbb{1}_{\{\tau_{j-1}^k \leq \theta < \tau_j^k\}} J_j\big(\theta, X_\theta^{t,x;\boldsymbol{u}}; \boldsymbol{u}\big) \prod_{i=k}^{j-1} \ell_i\big(X_{\tau_i^k}^{t,x;\boldsymbol{u}}\big)$$

---

[6]For a similar result in the context of SDEs, see [Kry09, Theorem 3.1.7, p. 132].

$$(10b) \qquad\qquad \leq \sum_{j=k}^{n+1} \mathbb{1}_{\{\tau_{j-1}^k \leq \theta < \tau_j^k\}} V_j\big(\theta, X_\theta^{t,x;\boldsymbol{u}}\big) \prod_{i=k}^{j-1} \ell_i\big(X_{\tau_i^k}^{t,x;\boldsymbol{u}}\big)$$

where (10a) and (10b) follow from Assumption 4.1.b.ii., and Lemma 3.2, respectively. In the light of the tower property of conditional expectation [Kal97, Theorem 5.1], arbitrariness of $\boldsymbol{u} \in \mathcal{U}_t$, and obvious inequality $V_j \leq V_j^*$, we arrive at (9a).

To prove (9b), we define a sequence of uniformly bounded upper semicontinuous functions $(\phi_j)_{j=k}^n \subset \mathrm{USC}(\mathbb{S})$ such that $\phi_j \leq V_{j*}$ on $\mathbb{S}$. Mimicking the ideas in the proof of our earlier work [MECL12, Theorem 4.7], it is possible to establish that given $\epsilon > 0$ for all $j \in \{k, \cdots, n\}$ there exists an admissible control policy $\boldsymbol{u}_j^\epsilon$ such that

$$(11) \qquad\qquad \phi_j(t,x) - 3\epsilon \leq J_j(t,x; \boldsymbol{u}_j^\epsilon) \qquad \forall (t,x) \in \mathbb{S}.$$

Let us fix $\boldsymbol{u} \in \mathcal{U}_t$ and $\epsilon > 0$, and define

$$(12) \qquad\qquad \boldsymbol{v}^\epsilon := \mathbb{1}_{[t,\theta]}\boldsymbol{u} + \mathbb{1}_{]\theta,T]} \sum_{j=k}^n \mathbb{1}_{\{\tau_{j-1}^k \leq \theta < \tau_j^k\}} \boldsymbol{u}_j^\epsilon,$$

where $\boldsymbol{u}_j^\epsilon$ satisfies (11). Notice that Assumption 4.1.a. guarantees $\boldsymbol{v}^\epsilon \in \mathcal{U}_t$. In accordance with (11), (12) and in light of Assumption 4.1.i., one obtains the tower property from (10a) as follows:

$$
\begin{aligned}
V_k(t,x) \geq J_k(t,x; \boldsymbol{v}^\epsilon) &= \mathbb{E}\bigg[\mathbb{E}\Big[\prod_{i=k}^n \ell_i\big(X_{\tau_i^k}^{t,x;\boldsymbol{v}^\epsilon}\big) \Big| \mathcal{F}_\theta \Big] \bigg] \\
&= \mathbb{E}\bigg[\sum_{j=k}^{n+1} \mathbb{1}_{\{\tau_{j-1}^k \leq \theta < \tau_j^k\}} J_j\big(\theta, X_\theta^{t,x;\boldsymbol{u}}; \boldsymbol{u}_j^\epsilon\big) \prod_{i=k}^{j-1} \ell_i\big(X_{\tau_i^k}^{t,x;\boldsymbol{u}}\big) \bigg] \\
&= \mathbb{E}\bigg[\sum_{j=k}^{n+1} \mathbb{1}_{\{\tau_{j-1}^k \leq \theta < \tau_j^k\}} \Big(\phi_j\big(\theta, X_\theta^{t,x;\boldsymbol{u}}\big) - 3\epsilon \Big) \prod_{i=k}^{j-1} \ell_i\big(X_{\tau_i^k}^{t,x;\boldsymbol{u}}\big) \bigg].
\end{aligned}
$$

Since V_{j*} is lower semicontinuous, in view of [Ren99, Lemma 3.5] one may pick a sequence of increasing continuous functions $(\phi_j^m)_{m \in \mathbb{N}}$ that converges point-wise to V_j. By boundedness of $(\ell_j)_{i=1}^n$ and the dominated convergence Theorem, we get

$$V_k(t,x) \geq \mathbb{E}\bigg[\sum_{j=k}^{n+1} \mathbb{1}_{\{\tau_{j-1}^k \leq \theta < \tau_j^k\}} \Big(V_{j*}\big(\theta, X_\theta^{t,x;\boldsymbol{u}}\big) - 3\epsilon \Big) \prod_{i=k}^{j-1} \ell_i\big(X_{\tau_i^k}^{t,x;\boldsymbol{u}}\big) \bigg]$$

Since $\boldsymbol{u} \in \mathcal{U}_t$ and $\epsilon > 0$ are arbitrary, this implies the assertion (9b). $\square$

Remark 4.6. Theorem 4.5 introduces DPP's in a weaker sense than the standard DPP in stochastic optimal control problems [FS06]. Namely, one does not need to verify the measurability of the value functions V_k in (4) so as to apply the DPP's. Notice that in general this measurability issue is non-trivial due to the supremum operation running over possibly uncountably many policies.

5. Applications to Controlled Diffusions

The objective of this section is to demonstrate how the weak DPP derived in §4 adapts to the context of controlled diffusion processes. This application results in a series of Hamilton-Javobi-Bellman PDE's, where each PDE is understood in the discontinuous viscosity sense with some boundary conditions both in viscosity and Dirichlet (pointwise) senses. To this end, we shall first introduce formally the standard probability space setup for SDEs, then proceed with some preliminaries so as to pave the ground for the required Assumptions 4.1 to hold. The

section consists of subsections concerning PDE derivation and boundary conditions along with further discussions so as to deploy the existing PDE solvers to numerically compute our PDE characterization.

Let Ω be $\mathcal{C}\big([0,T],\mathbb{R}^{z_d}\big)$, the set of continuous functions from $[0,T]$ into $\mathbb{R}^{z_d}$, and let $(z_t)_{t\geq 0}$ be the canonical process, i.e., $z_t(\omega) := \omega_t$. We consider $\mathbb{P}$ as the Wiener measure on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F})$, where $\mathbb{F}$ is the smallest right continuous filtration on Ω such that the process $(z_t)_{t\geq 0}$ is adapted to. Let us recall that $\mathbb{F}_t := (\mathcal{F}_{t,s})_{s\geq 0}$ is the auxiliary subfiltration defined as $\mathcal{F}_{t,s} := \sigma(z_r - z_t, t \leq r \leq t \vee s)$

Let $\mathbb{U} \subset \mathbb{R}^{d_u}$ be a control set, and $\mathcal{U}_t$ denote the set of all $\mathbb{F}_t$- progressively measurable mappings into $\mathbb{U}$. For every $\boldsymbol{u} = (u_t)_{t\geq 0}$ we consider the following $\mathbb{R}^d$-valued SDE[7]:

$$(13) \qquad \mathrm{d}X_s = f(X_s, u_s)\,\mathrm{d}s + \sigma(X_s, u_s)\,\mathrm{d}W_s, \qquad X_t = x, \quad s \geq t,$$

where $f : \mathbb{R}^d \times \mathbb{U} \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^d \times \mathbb{U} \to \mathbb{R}^{d\times d_z}$. are measurable maps, and $W_s := z_s$ is the canonical process.

Assumption 5.1. *We stipulate that*

a. $\mathbb{U} \subset \mathbb{R}^m$ *is compact;*
b. *f and σ are continuous and Lipschitz in its first argument uniformly with respect to the second;*
c. *The diffusion term σ of the SDE (13) is uniformly non-degenerate, i.e., there exists $\delta > 0$ such that for all $x \in \mathbb{R}^d$ and $u \in \mathbb{U}$, $\|\sigma\sigma^\top\| > \delta$.*

It is well-known [Bor05] that under Assumptions 5.1.a. and 5.1.b. there exits a unique strong solution to the SDE (13); let us denote it by $\big(X_s^{t,x;\boldsymbol{u}}\big)_{s\geq t}$. For future notational simplicity, we slightly modify the definition of $X_s^{t,x;\boldsymbol{u}}$, and extend it to the whole interval $[0,T]$ where $X_s^{t,x;\boldsymbol{u}} := x$ for all s in $[0,t]$.

In addition to Assumptions 5.1 on the SDE (13), we impose the following assumption on the motion planning sets that allows us to guarantee the continuity of sequential exit-times as required for the DPP obtained in the preceding section.

Assumption 5.2 (Exterior Cone Condition)**.** *The open sets $(A_i)_{i=1}^n \subset \mathfrak{B}(\mathbb{R}^d)$ satisfy the following condition: for every $i \in \{1, \cdots, n\}$, there are positive constants h, r an $\mathbb{R}^d$-value bounded map $\eta : A_i^c \to \mathbb{R}^n$ such that*

$$\mathrm{B}_{rt}(x + \eta(x)t) \subset A_i^c \qquad \text{for all } x \in A_i^c \text{ and } t \in (0, h]$$

where $\mathrm{B}_r(x)$ denotes an open ball centered at x and radius r and A_i^c stands for the complement of the set A_i.

Remark 5.3. If the set A_i is bounded and its boundary ∂A_i is smooth, then Assumption 5.2 holds. Furthermore, boundaries with corners may also satisfy Assumption 5.2; Figure 5 depicts two different examples.

5.1. **Sequential Partial Differential Equations.** This subsection establishes a connection between the DPP introduced in Theorem 4.5 and a sequence of PDEs, all of the latter are meant in the sense of discontinuous viscosity solutions; for the general theory of viscosity solutions we refer to [CIL92] and [FS06]. For numerical solutions to these PDEs, one also needs some boundary conditions, and that will be the objective of the next subsection.

---

[7]We slightly abuse notation and earlier used σ for the sigma algebra as well. However, it will be always clear from the context to which σ we refer.

(a) Exterior cone condition holds at every point of the boundary.

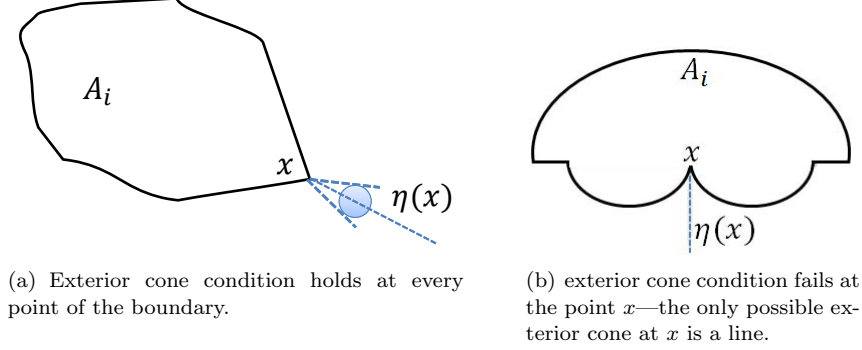(b) exterior cone condition fails at the point x—the only possible exterior cone at x is a line.

FIGURE 5. Exterior cone condition of the boundary

To apply the proposed DPP, one has to make sure that Assumptions 4.1 are satisfied. As pointed out in Remark 4.2, the only nontrivial assumption in the context of SDEs is Assumption 4.1.b.iii. The following proposition addresses this issue, and allows us to employ the DPP of Theorem 4.5 for the main result of this subsection.

Proposition 5.4. *Consider the SDE* (13) *where Assumptions 5.1 holds. Suppose that the open sets* $(A_i)_{i=1}^n \subset \mathfrak{B}(\mathbb{R}^d)$ *satisfy the exterior cone condition in Assumption 5.2. Let* $\left(\Theta_i^{A_{1:n}}\right)_{i=1}^n$ *be the respective sequential exit-times as defined in Definition 3.1. Given intermediate times* $(T_i)_{i=1}^n$ *and control policy* $\boldsymbol{u} \in \mathcal{U}_t$, *for any* $i \in \{1.\cdots, n\}$, *initial condition* $(t, x) \in \mathbb{S}$, *and sequence of initial conditions* $(t_m, x_m) \to (t, x)$, *we have*

$$\lim_{m \to \infty} \tau_i(t_m, x_m) = \tau_i(t, x) \quad \mathbb{P}\text{-a.s.}, \qquad \tau_i(t, x) := \Theta_i^{A_{1:n}}(t, x) \wedge T_i.$$

Moreover, one can show that the above result readily leads to the continuity of the stochastic mapping $(t, x) \mapsto X_{\tau_i(t,x)}^{t,x;\boldsymbol{u}}$ *with probability one, i.e.,* $\lim_{m \to \infty} X_{\tau_i(t_m,x_m)}^{t_m,x_m;\boldsymbol{u}} = X_{\tau_i(t,x)}^{t,x;\boldsymbol{u}}$ $\mathbb{P}$*-a.s. for all* i.

Proof. See Appendix B. $\square$

Definition 5.5 (Dynkin Operator)**.** *Given* $u \in \mathbb{U}$, *we denote by* $\mathcal{L}^u$ *the Dynkin operator (also known as the infinitesimal generator) associated to the controlled diffusion* (13) *as*

$$\mathcal{L}^u \Phi(t, x) := \partial_t \Phi(t, x) + f(x, u).\partial_x \Phi(t, x) + \frac{1}{2} Tr[\sigma \sigma^\top(x, u)\partial_x^2 \Phi(t, x)],$$

where Φ *is a real-valued function smooth on the interior of* $\mathbb{S}$, *with* $\partial_t \Phi$ *and* $\partial_x \Phi$ *denoting the partial derivatives with respect to* t *and* x *respectively, and* $\partial_x^2 \Phi$ *denoting the Hessian matrix with respect to* x. *We refer to* [Kal97, Theorem 17.23] *for more details on the above differential operator.*

Theorem 5.6 is the main result of this subsection, which provides a characterization of the value functions V_k in terms of Dynkin operator in Definition 5.5 in the interior of the set of interest, i.e., $[0, T_k[\times A_k$. This result is a direct consequence of the DPP in Theorem 4.5, and Itô's formula; for similar technique we refer to the proof of [MECL12, Theorem 4.10].

Theorem 5.6. *Consider the system* (13), *and suppose that Assumptions 5.1 hold. Let the value functions* $V_k : \mathbb{S} \to \mathbb{R}^d$, $k = 1, \ldots, n$ *be as defined in* (7), *where the sets* $(A_i)_{i=1}^n$ *satisfy the Assumption 5.2, and the payoff functions* $(\ell_i)_{i=1}^n$ *are all lower semicontinuous. Then,*

○ *the lower semicontinuous envelope of V_k is a viscosity supersolution of*

$$-\sup_{u\in\mathbb{U}}\mathcal{L}^u V_{k*}(t,x)\geq 0 \qquad on \quad [0,T_k[\times A_k;$$

○ *the upper semicontinuous envelope of V is a viscosity subsolution of*

$$-\sup_{u\in\mathbb{U}}\mathcal{L}^u V_k^*(t,x)\leq 0 \qquad on \quad [0,T_k[\times A_k.$$

Proof. We refer to Appendix B for a sketch of proof and [MECL12, Theorem 4.10] for a detailed analysis of the same technique. □

5.2. **Boundary Conditions.** To numerically solve the PDE characterization of the previous part, one needs some boundary value conditions on the complement set of the PDE, which is addressed in the following:

Proposition 5.7. *Suppose that the hypotheses of Theorem 5.6 hold. Then the value functions V_k introduced in (7) satisfy the following boundary value conditions:*

(14a) $$V_k(t,x) = V_{k+1}(t,x)\ell_k(x) \qquad on \quad [0,T_k]\times A_k^c\bigcup\{T_k\}\times\mathbb{R}^d$$

(14b) $$V_k^*(t,x) \leq V_{k+1}^*(t,x)\ell_k^*(x) \qquad on \quad [0,T_k]\times A_k^c\bigcup\{T_k\}\times\mathbb{R}^d$$

Proof. See Appendix B, along with a preparatory lemma. □

Proposition 5.7 provides boundary conditions for the value function V_k not only in Dirichlet (pointwise) (14a) sense but also in the discontinuous viscosity sense (14b). Observe that the value functions V_k are all lower semicontinuous since each of them is the supremum over family of lower semicontinuous functions J_k, see Lemma B.1. It then follows that the pointwise boundary condition (14a) indicates the other side of viscosity boundary condition of (14b), i.e., $V_{k*}(t,x)\geq V_{k+1*}(t,x)\ell_{k*}(x)$.

5.3. **Discussion on Numerical Issues.** The objective of the current section was to propose an alternative characterization of the value functions $V_k:\mathbb{S}\to\mathbb{R}$ in (7) for the case of controlled diffusion processes governed via the SDE (13). To this end, §5.1 developed a PDE characterization of the value function V_k within the set $[0,T_k[\times A_k$ along with some boundary conditions in terms of the successor value function V_{k+1} provided in §5.2. Given value function V_{k+1}, to obtain the value function V_k, the only non-trivial set of initial conditions becomes $[0,T_k[\times A_k$ for which it is required to solve a certain PDE with some boundary (possibly both terminal and lateral) conditions. Since $V_{n+1}\equiv 1$, one can infer that Theorem 5.6 and Proposition 5.7 suggest a series of PDE equations for which the first one has known boundary condition ℓ_n, while the boundary conditions of the subsequent steps are determined by the solution of the preceding PDE step, i.e., V_{k+1} provides boundary conditions for the PDE corresponding to the value function V_k. Let us highlight once again that the basic motion planning maneuver involving only two sets is effectively the same as the first step of this series of PDEs and was studied in our earlier work [MECL12].

Before proceeding to apply the PDE characterization of this section to obtain the desired motion-planning initial sets introduced in Definition 2.6, we need to properly justify the following two concerns:

(i) On the one hand, for the definition (2a) we need to assume that the goal set G_i is closed so as to allow continuous transition into G_i; see Assumption 2.2 and the discussion preceding it. On the other hand, in order to invoke the DPP argument of §4 and its consequent PDE

in §5.1, we need to impose that the payoff functions $(\ell_i)_{i=1}^n$ are all lower semicontinuous; see Assumption 4.1.c. In the case of the value function V in (4a), this constraint results in $(G_i)_{i=1}^n$ all being open which is obviously in contradiction to our earlier assumption in accordance to (2a).

(ii) Due to the numerical issues, most of existing PDE solvers provide theoretical guarantees for *continuous* viscosity solutions, e.g., [Mit05], whereas our characterization is indeed in discontinuous form. Therefore, it is a natural question whether we could employ the existing off-the-shelf toolbox to numerically calculate our desired value function.

Let us initially highlight the following points: Concerning (i) it should be mentioned that this contradiction is not applicable for the motion-planning initial set (2b) since the goal set G_i can be simply chosen to be open without confining the continuous transitions. Concerning (ii), we would like to stress that this discontinuous formulation is inevitable since the value functions defined in (4) are in general discontinuous, and any PDE approach has to rely on discontinuous versions.

Moreover, we next propose an ϵ-conservative but precise way of characterizing the motion-planning initial set of (2a) as well as justifying the numerical concern in (ii) to employ the existing off-the-shelf PDE solvers. Given $(W_i, G_i) \in \mathfrak{B}(\mathbb{R}^d) \times \mathfrak{B}(\mathbb{R}^d)$, let us construct a smaller goal set $G_i^\epsilon \subset G_i$ such that $G_i^\epsilon := \{x \in G_i \mid \operatorname{dist}(x, G_i^c) \geq \epsilon\}$.[8] For sufficiently small $\epsilon > 0$ one may observe that $W_i \setminus G_i^\epsilon$ satisfies Assumption 5.2. Note that this is always possible if $W_i \setminus G_i$ satisfies Assumption 5.2 since one can simply take $\epsilon < h/2$, where h is as defined in Assumption 5.2. Figure 6 depicts this situation.
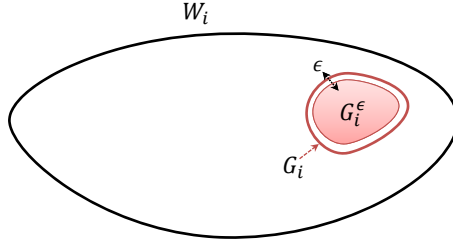

FIGURE 6. Construction of the sets G_i^ϵ from G_i as described in §5.3

Formally we define the payoff function $\ell_i^\epsilon : \mathbb{R}^d \to \mathbb{R}$ as follows:

$$\ell_i^\epsilon(x) := \left(1 - \frac{\operatorname{dist}(x, G_\epsilon)}{\epsilon}\right) \vee 0.$$

Replacing the goal sets G_i^ϵ and payoff functions ℓ_i^ϵ in (4a), we arrive at the value functions

$$V^\epsilon(t, x) := \sup_{\boldsymbol{u} \in \mathcal{U}_t} \mathbb{E}\left[\prod_{i=1}^n \ell_i^\epsilon(X_{\eta_i^\epsilon}^{t,x;\boldsymbol{u}})\right], \qquad \eta_i^\epsilon := \Theta_i^{B_{1:n}^\epsilon} \wedge T, \qquad B_i^\epsilon := W_i \setminus G_i^\epsilon.$$

It is straightforward to inspect that $V^\epsilon \leq V$ since the goal sets are smaller with respect to the actual goal sets G_i. Moreover, one can show that $V(t, x) = \lim_{\epsilon \downarrow 0} V^\epsilon(t, x)$ on the set $(t, x) \in [t, T[\times \mathbb{R}^d$, which indicates that the approximation scheme can be arbitrarily precise. Note that the approximated payoff functions ℓ_i^ϵ are, by construction, Lipschitz continuous that in light of uniform continuity of the process, Lemma B.1, yields to the continuity of the value function V^ϵ. Hence, the discontinuous PDE characterization of §5.1 is simplified to continuous

---

[8]$\operatorname{dist}(x, A) := \inf_{y \in A} \|x - y\|$, where $\|\cdot\|$ stands for the Euclidean norm.

regime, for which there exist PDE solvers for numerical computations. We refer to [MECL12, Section 5] for a detailed analysis of the proposed approximation scheme.

6. Numerical Example: Chemical Langevin Equation for a Biological Switch

In this section we apply the theoretical results in the preceding sections to a biological switch network. Multistable biological systems are often seen in nature [BSS01]. The inherent stochasticity in these systems can be substantial—they influence convergence, or may even lead to switching behavior from one equilibrium to another. The pioneer works on modeling reactions in biochemical network are based on countable state Markov chains, which describe the evolution of molecular numbers. Due to the Markov property of chemical reactions, one can track the time evolution of the probability distribution for molecular populations as a family of ordinary differential equations called the *chemical master equation* (CME) [AGA09, ESKPG05]; it is also known as the forward Kolmogorov equation.

An SDE approach toward modeling the stochastic molecular numbers of species has been proposed in the literature. It is, of course, assumed that a molecular species number may take non-integer numbers. This assumption is usually reasonable for large molecular populations, whereas for low copy numbers, it may not be reliable—a small number, say, 0.1 protein copies can lead to an entirely different dynamic behavior than one would observe from exactly 0 copies. The time-continuity of stochastic processes makes the analysis significantly more tractable than the CME. In this method, one can approximate the molecular numbers via continuous time Markov process, where the latter is an approximation of the jump Markov process that underlines the CME. This stochastic continuous-time approximation is called the *chemical Langevin equation* or the *diffusion approximation*, see [Kha] and the reference therein for further details on modeling and analysis of stochastic biochemical networks.

Another difficulty with the diffusion approximation is that the model is typically not well-posed since it may assign a negative number to a molecular species. Nevertheless, this issue can be neglected when the focus of observation is away from low numbers of each species. In this section we consider the following chemical Langevin formulation of a two gene network:

$$(15) \quad \begin{cases} dX_t = \big(f(Y_t, \boldsymbol{u}_x) - \mu_x X_t\big)dt + \sqrt{f(Y_t, \boldsymbol{u}_x)}dW_t^1 + \sqrt{\mu_x X_t}dW_t^2, & X_0 = x_0 \\ dY_t = \big(g(X_t, \boldsymbol{u}_y) - \mu_y Y_t\big)dt + \sqrt{g(X_t, \boldsymbol{u}_y)}dW_t^3 + \sqrt{\mu_y Y_t}dW_t^4, & Y_0 = y_0 \end{cases}$$

where X_t and Y_t are the concentration of the two repressor proteins with the respective degradation rates μ_x and μ_y; $(W_t^i)_{t \geq}$ are independent standard Brownian motion processes. Functions f and g are repression functions that describe the impact of each protein on the other's rate of synthesis controlled via some external inputs $\boldsymbol{u}_x$ and $\boldsymbol{u}_y$.

In the absence of exogenous control signals, the authors of [Che00] study sufficient conditions on the drifts f and g under which the system dynamic (15) without the diffusion term has two (or more) stable equilibria. In this case, system (15) can be viewed as a biological switch network. The aforementioned theoretical results are also experimentally investigated in [GCC00] for a genetic toggle switch in *Escherichia coli*.

In this article we consider the biological switch dynamics such that the degradation rates of proteins are influenced by some external control signals. The practical feasibility of this assumption has successfully been experimented in [MSSO$^+$11]. The level of repression is described by a *Hill* function, which models cooperativity of binding as follows:

$$f(y, u) := \frac{\theta_1^{n_1} k_1}{y^{n_1} + \theta_1^{n_1}} u, \qquad g(x, u) := \frac{\theta_2^{n_2} k_2}{x^{n_2} + \theta_2^{n_2}} u,$$

where θ_i are the threshold of the production rate with respective exponents n_i, and k_i are the production scaling factors. The parameter u represents the role of external signals that affect the production rates, for which the control sets are $\mathbb{U}_x := [\underline{u}_x, \overline{u}_x]$ and $\mathbb{U}_y := [\underline{u}_y, \overline{u}_y]$ with nominal value $\hat{u} := 1$. As explained in [Che00], the *nullclines* of the system are $\frac{f(y,\hat{u})}{\mu_1}$ and $\frac{g(x,\hat{u})}{\mu_2}$, which determine whether the system has multiple stable equilibria. In this example we consider system (15) with the following parameters: $\theta_i = 40$, $\mu_i = 0.04$, $k_i = 4$ for both $i \in \{1, 2\}$, and exponents $n_1 = 4$, $n_2 = 6$. Figure 7(a) depicts the drift nullclines and the equilibria of the system. The equilibria z_a and z_c are stable, while z_b is the unstable one. We should remark that the "stable equilibrium" of SDE (15) is understood in the absence of the diffusion term, as the noise may very well push the states from one stable equilibrium to another.

(a) Nullclines and equilibria of the drift of the SDE (15); x_a and x_b are stable, and x_c is unstable.

(b) The set A is an avoidance region contained in the region of attraction of the stable equilibria x_a and x_c, B is the target set around the unstable equilibrium x_b, and C is the maintenance margin.
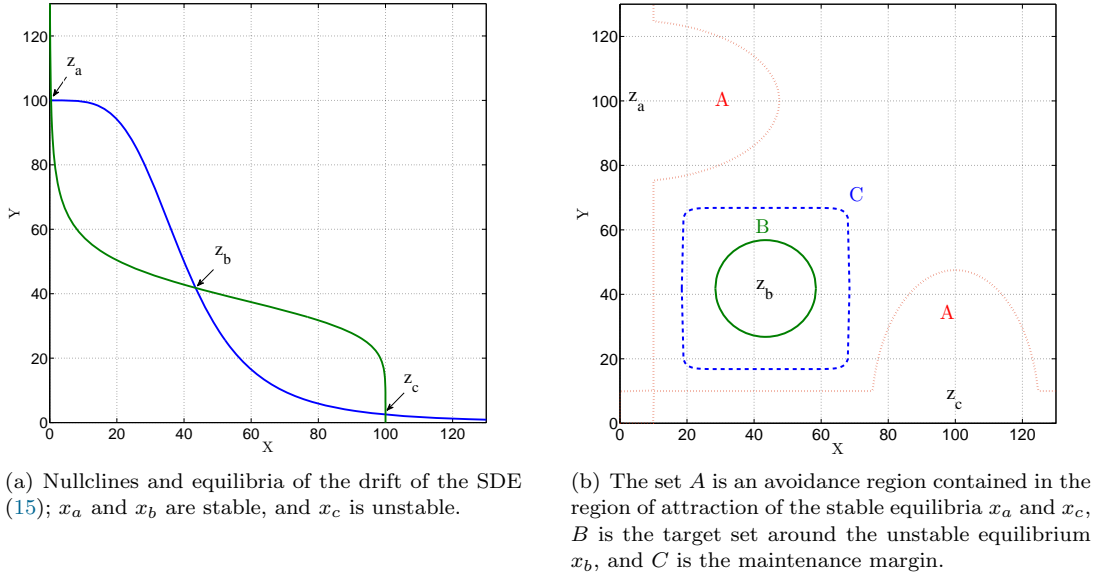
FIGURE 7. State space of the biological switch (15) with desired motion planning sets.

In this simulation, we first aim to steer the number of proteins towards a target set around the unstable equilibrium by synthesizing appropriate input signals $\boldsymbol{u}_x$ and $\boldsymbol{u}_y$ within a certain time horizon, say T_1. During this task we opt to avoid the region of attraction of the stable equilibria as well as low numbers for each protein; the latter justifies our model being well-posed in the region of interest. The aforementioned target and avoid sets are denoted, respectively, by the closed sets B and A in Figure 7(b). In the second phase of the task, once the trajectory visits the target set B, it is required to keep the molecular populations within a slightly larger margin around equilibria for some time, say T_2; Figure 7(b) depicts this maintenance margin by the open set C. In the context of reachability, the second phase is known as *viability* [Aub91]; for an almost-sure stochastic counterpart see for instance [AD90, AP98].

Let us highlight two technical points here: (i) set C must be chosen strictly larger than B in order to allow the process to maneouver inside the interior of the maintenance set C once it hits the target set B. From a theoretical standpoint, this is necessary since the probability of hitting set B and remaining inside B for all future has zero probability; this is due to the non-degenerate property of SDE (15), see the proof of Proposition 5.4 for a rigorous analysis of this issue. (ii) Since the process is non-degenerate, in principle the noise can push the process

anywhere in the long-run with positive probability. This fact indicates that the probability of success in the second phase of the motion planning task decreases with the time horizon T_2, and it tends to zero as T_2 goes to ∞.

Therefore, the motion planning consists of two parts: reaching the target set B while avoiding the set A within the certain horizon T_1, and staying in the set C for the certain time T_2 after visiting the set B for the first time. In view of motion-planning events introduced in Definition 2.1, the first phase of the path can be expressed as $(A^c \rightsquigarrow B)_{\leq T_1}$, and the second phase as $(C \xrightarrow{T_2} C)$; see (1) for detailed definitions of these symbols. By defining the joint process $Z_{\cdot}^{t,z;\boldsymbol{u}} := \big[X_{\cdot}^{t,x;\boldsymbol{u}}, Y_{\cdot}^{t,y;\boldsymbol{u}} \big]$, with the initial condition $z := (x,y)$, the desired excursion is a combination of the events studied in the preceding sections and, with a slight abuse of notation, can be expressed by

$$\left\{ Z_{\cdot}^{t,z;\boldsymbol{u}} \models (A^c \rightsquigarrow B)_{\leq T_1} \circ (C \xrightarrow{T_2} C) \right\}.$$

The above event depends, of course, on the initial condition (t,x) and control policy $\boldsymbol{u}$, and the objective is to maximize its probability over all admissible policies $\boldsymbol{u} := [\boldsymbol{u}_x, \boldsymbol{u}_y]$. The desired path is not exactly in the framework of Definition 2.1 but, nonetheless, one can invoke the same ideas as in §3 and introduce the following value functions:

$$(16\text{a}) \qquad V_1(t,z) := \sup_{\boldsymbol{u} \in \mathcal{U}_t} \mathbb{E}\Big[\mathbb{1}_B\big(Z_{\tau_1^1}^{t,z;\boldsymbol{u}} \big) \mathbb{1}_C\big(Z_{\tau_2^1}^{t,z;\boldsymbol{u}} \big) \Big],$$

$$(16\text{b}) \qquad V_2(t,z) := \sup_{\boldsymbol{u} \in \mathcal{U}_t} \mathbb{E}\Big[\mathbb{1}_C\big(Z_{\tau_2^2}^{t,z;\boldsymbol{u}} \big) \Big],$$

where τ_1^1 and τ_2^2 are defined in a same spirit as (7) with given sets $A_1 := (A \cup B)^c$ and $A_2 := C$. However, the stopping time τ_2^1 requires a slight modification so as to address the combination of both motion-planning events introduced in Definition 2.1: $\tau_2^1 := \Theta_2^{A_{1:2}} \wedge (\tau_1^1 + T_2)$.

The solution of our motion planning objective is the value function V_1 in (16a), which in view of Theorem 5.6 is characterized by the Dynkin differential operator in the interior of $[0, T_1[\times (A \cup B)^c$. However, we need first to solve numerically for the auxiliary value function V_2 in (16b) in order to provide boundary conditions for the PDE corresponding to V_1 by

$$(17) \qquad V_1(t,z) = \mathbb{1}_B(z) V_2(t,z), \qquad (t,x) \in [0,T_1] \times (A \cup B) \bigcup \{T_1\} \times \mathbb{R}^n.$$

It is straightforward to observe that the boundary condition for the value function V_2 is

$$V_2(t,z) = \mathbb{1}_C(z), \qquad (t,x) \in [0, T_1 + T_2] \times C^c \bigcup \{T_1 + T_2\} \times \mathbb{R}^n.$$

Therefore, we need to solve the PDE of V_2 with the above boundary condition backward from the time $T_1 + T_2$ to the time T_1, and then at time T_1 restrict the value function V_2 onto the set B to provide boundary conditions for the value function V_1. Thus, the value function V_1 can be computed via solving the same PDE from T_1 to 0 but along with different boundary conditions provided by the preceding step. According to Definition 5.5 for any smooth function $\phi := \phi(t,x,y)$ the Dynkin operator $\mathcal{L}^u$ can be simplified to

$$\sup_{u \in \mathbb{U}} \mathcal{L}^u \phi(t,x,y)$$

$$= \max_{u \in \mathbb{U}} \Big[\partial_t \phi + \partial_x \phi \big(f(y, u_x) - \mu_x x \big) + \partial_y \phi \big(g(x, u_y) - \mu_y y \big)$$

$$+ \frac{1}{2} \partial_x^2 \phi \big(f(y, u_x) + \mu_x x \big) + \frac{1}{2} \partial_y^2 \phi \big(g(x, u_y) + \mu_y y \big) \Big]$$

$$= \partial_t \phi - \big(\partial_x \phi - \frac{1}{2} \partial_x^2 \phi \big) \mu_x x - \big(\partial_y \phi - \frac{1}{2} \partial_y^2 \phi \big) \mu_y y$$

$$+ \max_{u_x \in [\underline{u}_x, \overline{u}_x]} \left[f(y, u_x) \left(\partial_x \phi + \frac{1}{2} \partial_x^2 \phi \right) \right] + \max_{u_y \in [\underline{u}_y, \overline{u}_y]} \left[g(x, u_y) \left(\partial_y \phi + \frac{1}{2} \partial_y^2 \phi \right) \right].$$

On account of Theorem 5.6 and linearity of the drift terms in u, one can propose an optimal policy in terms of derivatives of the value functions V_1 and V_2 in (16), respectively, for the first and second phase of the motion:

$$\boldsymbol{u}_x^*(t, x, y) = \begin{cases} \overline{u}_x(t, x, y) & \text{if} \quad \partial_x V_i(t, x, y) + \frac{1}{2} \partial_x^2 V_i(t, x, y) \geq 0, \\ \underline{u}_x(t, x, y) & \text{if} \quad \partial_x V_i(t, x, y) + \frac{1}{2} \partial_x^2 V_i(t, x, y) < 0, \end{cases}$$

$$\boldsymbol{u}_y^*(t, x, y) = \begin{cases} \overline{u}_y(t, x, y) & \text{if} \quad \partial_y V_i(t, x, y) + \frac{1}{2} \partial_y^2 V_i(t, x, y) \geq 0, \\ \underline{u}_y(t, x, y) & \text{if} \quad \partial_y V_i(t, x, y) + \frac{1}{2} \partial_y^2 V_i(t, x, y) < 0, \end{cases}$$

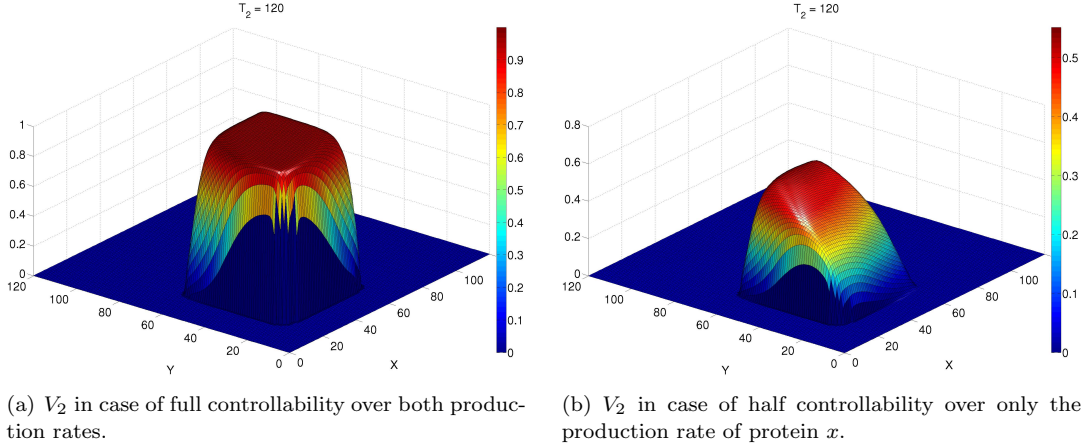where $i \in \{1, 2\}$ corresponds to the phase of the path.

(a) V_2 in case of full controllability over both production rates.

(b) V_2 in case of half controllability over only the production rate of protein x.

FIGURE 8. The value function V_2 as defined in (16b) corresponding to probability of staying in C for 120 time units.

In the sequel we investigate two scenarios: first, when full control over both production rates is possible, i.e., $\underline{u}_x = \underline{u}_y = 0$ and $\overline{u}_x = \overline{u}_y = 2$; second, when we only have access to the production rate of protein x, i.e., $\overline{u}_y = \underline{u}_y = \hat{u}$. Figure 8 depicts the probability distribution of staying in set C within the time horizon $T_2 = 120$ time units [9] in terms of the initial conditions $(x, y) \in \mathbb{R}^2$. V_2 is zero outside set C, as the process has obviously left C if it starts outside it. Figures 8(a) and 8(b) demonstrate the first and second control scenarios, respectively. Note that in the second case the probability of success dramatically decreases in comparison with the first. This result indicates the importance of full controllability of the production rates for the achievement of the given control objective.

Figure 9 depicts the probability of successively reaching set B within the time horizon $T_1 = 60$ time units and staying in set C for $T_2 = 120$ time units thereafter. Since the objective is to avoid set A, the value function V_1 takes zero value on A. Figures 9(a) and 9(b) demonstrate the first and second control scenarios, respectively. It is easy to observe the non-smooth behavior of the value function V_1 on the boundary of set B in Figure 9(b). This is indeed a consequence of the boundary condition explained in (17). All simulations in this subsection were obtained using the Level Set Method Toolbox [Mit05] (version 1.1), with a grid 121×121 in the region of simulation.

---

[9] Notice that the half-life of each protein is assumed to be 17.32 time units

(a) V_1 in case of full controllability over the production rates.

(b) V_1 in case of inability to increase the production rate of protein y.
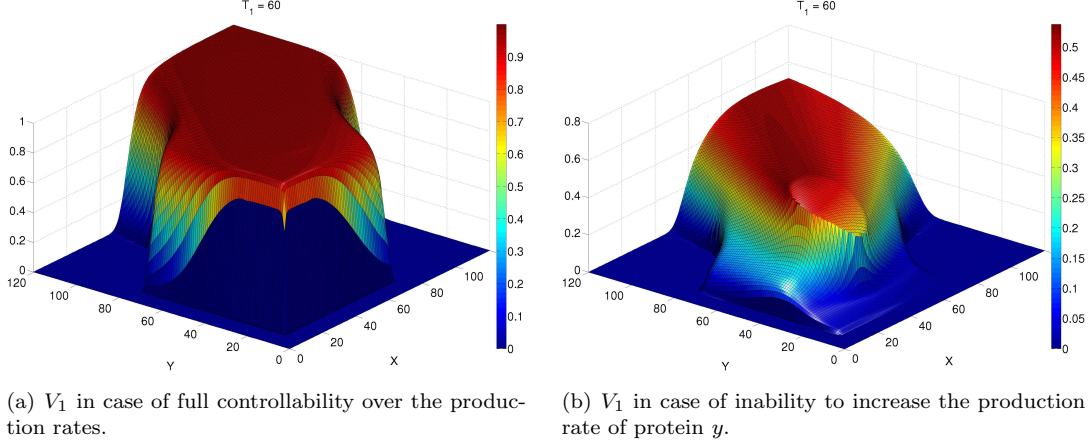
FIGURE 9. The value function V_1 as defined in (16a) corresponding to probability of staying in C for 120 time units, once it reaches B while avoiding A within 60 time units.

7. CONCLUSION AND FUTURE DIRECTIONS

In this article we introduced different notions of stochastic motion planning problems, with RCLL sample-paths realizations. Based on a class of stochastic optimal control problems, we proposed an alternative characterization of the set of initial conditions from which there exists an admissible policy to execute the desired maneuver with probability at least as much as some pre-specified value. We then established a weak DPP, which does not need to verify the measurability of the value functions, in terms of some auxiliary value functions. Subsequently, we focused on a case of diffusions as the solution of an SDE, and investigated the required conditions to apply the proposed DPP in the preceding section. It turned out that invoking the DPP one can solve a series of PDEs in a recursive fashion so as to numerically compute the desired initial set as well as admissible policy for the motion planning specifications. Finally, the performance of the proposed stochastic motion-planning notions was illustrated for a biological switch network.

For future work, in light of Theorem 4.5 which is in fact developed for RCLL processes, we aim to study the required conditions of the proposed DPP, Assumptions 4.1, for a larger class of stochastic processes, e.g., controlled Markov jump-diffusions.

APPENDIX A.

This appendix collects the missing proofs of the results presented in §3.

Proof of Proposition 3.6. We first show (5). Observe that it suffices to prove that

$$(18) \qquad \left\{ X_{\cdot}^{t,x;\boldsymbol{u}} \models \left[(W_1 \rightsquigarrow G_1) \circ \cdots \circ (W_n \rightsquigarrow G_n) \right]_{\leq T} \right\} = \bigcap_{i=1}^{n} \left\{ X_{\eta_i}^{t,x;\boldsymbol{u}} \in G_i \right\}$$

for all initial conditions (t, x) and policies $\boldsymbol{u}$, where the stopping time η_i is as defined in (4a). Let ω belong to the left-hand side of (18). In view of the definition (1a), there exists a set of instants $(s_i)_{i=1}^{n} \subset [t, T]$ such that for all i, $X_{s_i}^{t,x;\boldsymbol{u}}(\omega) \in G_i$ while $X_r^{t,x;\boldsymbol{u}}(\omega) \in W_i \setminus G_i =: B_i$ for all $r \in [s_{i-1}, s_i[$, where we set $s_0 = t$. It also follows by an induction argument that

$\eta_i(\omega) = \Theta_i^{B_{1:n}} = s_i$, which immediately leads to $X_{\eta_i(\omega)}^{t,x;\boldsymbol{u}}(\omega) \in G_i$ for all $i \le n$. This proves the relation " $\subset$ " between the left- and right-hand sides of (18). Now suppose that ω belongs to the right-hand side of (18). Then, for all $i \le n$ we have $X_{\eta_i(\omega)}^{t,x;\boldsymbol{u}}(\omega) \in G_i$. In view of the definition of stopping times η_i in (4a), it follows that $X_r^{t,x;\boldsymbol{u}}(\omega) \in B_i := W_i \setminus G_i$ for all $r \in [\eta_{i-1}(\omega), \eta_i(\omega)[$. Introducing the time sequence $s_i := \eta_i(\omega)$ implies the relation " $\supset$ " between the left- and right-hand sides of (18). Together with preceding argument, this implies (18).

To prove (6) we only need to show that

$$(19) \qquad \left\{ X_\cdot^{t,x;\boldsymbol{u}} \models (W_1 \xrightarrow{T_1} G_1) \circ \cdots \circ (W_n \xrightarrow{T_n} G_n) \right\} = \bigcap_{i=1}^n \left\{ X_{\widetilde{\eta}_i}^{t,x;\boldsymbol{u}} \in G_i \cap W_i \right\}$$

for all initial conditions (t,x) and policies $\boldsymbol{u}$, where the stopping time $\widetilde{\eta}_i$ is introduced in (4b). To this end, let us fix $(t,x) \in \mathbb{S}$ and $\boldsymbol{u} \in \mathcal{U}_t$, and assume that ω belongs to the left-hand side of (19). By definition (1b), for all $i \le n$ we have $X_{T_i}^{t,x;\boldsymbol{u}}(\omega) \in G_i$ and $X_r^{t,x;\boldsymbol{u}}(\omega) \in W_i$ for all $r \in [T_{i-1}, T_i]$. By a straightforward induction, we see that $\widetilde{\eta}_i(\omega) = T_i$, and consequently $X_{\widetilde{\eta}_i(\omega)}^{t,x;\boldsymbol{u}}(\omega) \in G_i \cap W_i$ for all $i \le n$. This establishes the relation " $\subset$ " between the left- and right-hand sides of (19). Now suppose ω belongs to the right-hand side of (19). Then, for all $i \le n$ we have $X_{\widetilde{\eta}_i(\omega)}^{t,x;\boldsymbol{u}}(\omega) \in G_i \cap W_i$. By virtue of Fact 3.5 and an induction argument once again, Assumption 3.4 guarantees that $\widetilde{\eta}_i(\omega) = T_i$, and consequently it follows that $X_{T_i}^{t,x;\boldsymbol{u}}(\omega) \in G_i$ and $X_r^{t,x;\boldsymbol{u}}(\omega) \in W_i$ for all $r \in [T_{i-1}, T_i]$. This establishes the relation " $\supset$ " in (18), and the assertion (19) follows. $\square$

APPENDIX B.

This appendix contains missing proofs of §5.

Proof of Proposition (5.4). The key step in the proof relies on the two Assumptions 5.1.c. and 5.2. There is a classical result on non-degenerate diffusion processes indicating that if the process starts from the tip of a cone, then it enters the cone with probability one [RB98, Corollary 3.2, p. 65]. This hints at the possibility that the aforementioned Assumptions together with almost sure continuity of the strong solution of the SDE (13) result in the continuity of sequential exit-times $\Theta_i^{A_{1:n}}$ and consequently τ_i. In the following we shall formally work around this idea.

Let us assume that $t_m \le t$ for notational simplicity, but one can effectively follow similar arguments for $t_m > t$. By the definition of the SDE (13),

$$X_r^{t_m,x_m;\boldsymbol{u}} = X_t^{t_m,x_m;\boldsymbol{u}} + \int_t^r f\big(X_s^{t_m,x_m;\boldsymbol{u}}, u_s\big)\mathrm{d}s + \int_t^r \sigma\big(X_s^{t_m,x_m;\boldsymbol{u}}, u_s\big)\mathrm{d}W_s \qquad \mathbb{P}\text{-a.s.}$$

By virtue of [Kry09, Theorem 2.5.9, p. 83], for all $q \ge 1$ we have

$$\mathbb{E}\Big[\sup_{r \in [t,T]} \big\| X_r^{t,x;\boldsymbol{u}} - X_r^{t_m,x_m;\boldsymbol{u}} \big\|^{2q} \Big] \le C_1(q,T,K)\mathbb{E}\Big[\big\| x - X_t^{t_m,x_m;\boldsymbol{u}} \big\|^{2q} \Big]$$

$$\le 2^{2q-1} C_1(q,T,K)\mathbb{E}\Big[\| x - x_m \|^{2q} + \big\| x_m - X_t^{t_m,x_m;\boldsymbol{u}} \big\|^{2q} \Big],$$

whence, in light of [Kry09, Corollary 2.5.12, p. 86], we get

$$(20) \qquad \mathbb{E}\Big[\sup_{r \in [t,T]} \big\| X_r^{t,x;\boldsymbol{u}} - X_r^{t_n,x_n;\boldsymbol{u}} \big\|^{2q} \Big] \le C_2(q,T,K,\|x\|)\big(\| x - x_n \|^{2q} + |t - t_n|^q \big).$$

In the above inequalities, K is the Lipschitz constant of f and σ mentioned in Assumption 5.1.b.; C_1 and C_2 are constant depending on the indicated parameters. Hence, in view of Kolmogorov's continuity criterion [Pro05, Corollary 1 Chap. IV, p. 220], one may consider a version of the

stochastic process $X^{t,x;\boldsymbol{u}}$ which is continuous in (t,x) in the topology of uniform convergence on compacts. This leads to the fact that $\mathbb{P}$-a.s., for any $\epsilon > 0$, for all sufficiently large m,

$$X_r^{t_m,x_m;\boldsymbol{u}} \in \mathrm{B}_\epsilon\big(X_r^{t_0,x_0;\boldsymbol{u}}\big), \qquad \forall r \in [t_m, T], \tag{21}$$

where $\mathrm{B}_\epsilon(y)$ denotes the ball centered at y and radius ϵ. For simplicity, let us define the shorthand $\tau_i^m := \tau_i(t_m, x_m)$.[10] By the definition of τ_i and Definition 3.1, since the set A_i is open, we conclude that

$$\exists \epsilon > 0, \qquad \bigcup_{s \in [\tau_{i-1}^0, \tau_i^0[} \mathrm{B}_\epsilon(X_s^{t_0,x_0;\boldsymbol{u}}) \cap A_i^c = \emptyset \qquad \mathbb{P}\text{-a.s.} \tag{22}$$

By definition $\tau_0^0 := \tau_0(t_0, x_0) = t_0$. As an induction hypothesis, let us assume τ_{i-1}^0 is $\mathbb{P}$-a.s. continuous, and we proceed with the induction step. One can deduce that (22) together with (21) implies that $\mathbb{P}$-a.s. for all sufficiently large m,

$$X_r^{t_m,x_m;\boldsymbol{u}} \in A_i, \qquad \forall r \in [t_m, \tau_i^0[.$$

In conjunction with $\mathbb{P}$-a.s. continuity of sample paths, this immediately leads to

$$\liminf_{m \to \infty} \tau_i^m := \liminf_{m \to \infty} \tau_i(t_m, x_m) \geq \tau_i(t_0, x_0) \qquad \mathbb{P}\text{-a.s.} \tag{23}$$

On the other hand, as mentioned earlier, the Assumptions 5.1.c. and 5.2 imply that the set of sample paths that hit the boundary of A_i and do not enter the set is negligible [RB98, Corollary 3.2, p. 65]. Hence

$$\forall \delta > 0, \quad \exists s \in \big[\Theta_i^{A_{1:n}}(t_0, x_0), \Theta_i^{A_{1:n}}(t_0, x_0) + \delta\big[, \quad X_s^{t_0,x_0;\boldsymbol{u}} \in (A_i^c)^\circ \quad \mathbb{P}\text{-a.s.},$$

where $(A_i^c)^\circ$ denotes the interior of the set A_i^c. Hence, in light of (21), $\mathbb{P}$-a.s. there exists $\epsilon > 0$, possibly depending on δ, such that for all sufficiently large m we have

$$X_s^{t_m,x_m;\boldsymbol{u}} \in \mathrm{B}_\epsilon(X_s^{t_0,x_0;\boldsymbol{u}}) \subset A_i^c$$

Recalling the induction hypothesis, we note that in accordance with the definition of sequential stopping times $\Theta_i^{A_{1:n}}$, one can infer that $\Theta_i^{A_{1:n}}(t_m, x_m) \leq s < \Theta_i^{A_{1:n}}(t_0, x_0) + \delta$. From arbitrariness of δ and the definition of τ_i, this leads to

$$\limsup_{m \to \infty} \tau_i(t_m, x_m) := \limsup_{m \to \infty} \Big(\Theta_i^{A_{1:n}}(t_m, x_m) \wedge T_i\Big) \leq \tau_i(t_0, x_0) \qquad \mathbb{P}\text{-a.s.},$$

where in conjunction with (23), $\mathbb{P}$-a.s. continuity of the map $(t,x) \mapsto \tau_i(t,x)$ at (t_0, x_0) for any $i \in \{1, \cdots, n\}$ follows. The assertion follows by induction.

The continuity of the mapping $(t,x) \mapsto X_{\tau_i(t,x)}^{t,x;\boldsymbol{u}}$ follows immediately from the almost sure continuity of the stopping time $\tau_i(t,x)$ in conjunction with the almost sure continuity of the version of the stochastic process $X_\cdot^{t,x;\boldsymbol{u}}$ in (t,x); for the latter let us note again that Kolmogorov's continuity criterion guarantees the existence of such a version in light of (20). □

Proof of Theorem 5.6. Here we briefly sketch the proof in words, and refer the reader to [MECL12, Theorem 4.10] for detailed arguments concerning the same technology to prove the assertion of the Theorem. Note that any $\mathbb{F}_t$-progressively measurable policy $\boldsymbol{u} \in \mathcal{U}_t$ satisfies Assumptions 4.1.a.. It is a classical result [Øks03, Chap. 7] that the strong solution $X_\cdot^{t,x;\boldsymbol{u}}$ satisfies Assumptions 4.1.b.i. and 4.1.b.ii. Furthermore, Proposition 5.4 together with almost sure path-continuity of the strong solution guarantees Assumption 4.1.b.iii. Hence, having met all the required assumptions of Theorem 4.5, one can employ the DPP (9). Namely, to establish the assertion concerning the *supersolution*, for the sake of contradiction, one can assume that there exists $(t_0, x_0) \in [0, T_k[\times A_k$, and a smooth function ϕ dominated by the value function V_{k*} where $(V_{k*} - \phi)(t_0, x_0) = 0$, such that for some $\delta > 0$, $-\sup_{u \in \mathbb{U}} \mathcal{L}^u \phi(t_0, x_0) < -2\delta$. Since ϕ is

---

[10]This notation is only employed in this proof.

smooth, the map $(t, x) \mapsto \mathcal{L}^u \phi(t, x)$ is continuous. Therefore, there exist $u \in \mathbb{U}$ and $r > 0$ such that $B_r(t_0, x_0) \subset [0, T_k[\times A_k$ and $-\mathcal{L}^u \phi(t, x) < -\delta$ for all (t, x) in $B_r(t_0, x_0)$. Let us define the stopping time $\theta(t, x)$ as the first exit time of trajectory $X^{t, x; u}$ from the ball $B_r(t_0, x_0)$. Note that by continuity of solutions to (13), $t < \theta(t, x)$ $\mathbb{P}$- a.s. for all $(t, x) \in B_r(t_0, x_0)$. Therefore, selecting $r > 0$ sufficiently small so that $\theta < \tau_k$, and applying Itô's formula, we see that for all $(t, x) \in B_r(t_0, x_0)$, $\phi(t, x) < \mathbb{E}\big[\phi\big(\theta(t, x), X^{t, x; u}_{\theta(t, x)}\big)\big]$. Now it suffices to take a sequence $(t_m, x_m, V_k(t_m, x_m))_{m \in \mathbb{N}}$ converging to $(t_0, x_0, V_{k*}(t_0, x_0))$. For sufficiently large m we have $V(t_m, x_m) < \mathbb{E}\big[V_{k*}\big(\theta(t_m, x_m), X^{t_m, x_m; u}_{\theta(t_m, x_m)}\big)\big]$ which, in view of the fact that $\theta(t_m, x_m) < \tau_k \wedge T_k$, contradicts the DPP in (9a). The *subsolution* property is proved effectively in a similar fashion. □

In order to provide boundary conditions, in the discontinuous viscosity sense, for the PDE equation in Theorem 5.6 we need a preparatory preliminary lemma that contains a stronger assertion than Proposition 5.4.

Lemma B.1. *Suppose that the conditions of Proposition 5.4 hold. Given a sequence of control policies $(\boldsymbol{u_m})_{m \in \mathbb{N}} \subset \mathcal{U}$ and initial conditions $(t_m, x_m) \to (t, x)$, we have*

$$\lim_{m \to \infty} \left\| X^{t, x; \boldsymbol{u_m}}_{\tau_i(t, x)} - X^{t_m, x_m; \boldsymbol{u_m}}_{\tau_i(t_m, x_m)} \right\| = 0 \quad \mathbb{P}\text{-}a.s., \qquad \tau_i(t, x) := \Theta^{A_{1:n}}_i(t, x) \wedge T_i.$$

Note that Lemma B.1 is indeed a stronger statement than Proposition 5.4 as the desired continuity is required uniformly with respective to control policy. Let us highlight that the stopping times $\tau_i(t, x)$ and $\tau_i(t_m, x_m)$ are both effected by control policies $\boldsymbol{u_m}$. But nonetheless, the mapping $(t, x) \mapsto X^{t, x; \boldsymbol{u_m}}_{\tau_i}$ is almost surely continuous irrespective of the policies $(\boldsymbol{u_m})_{m \in \mathbb{N}}$. For the proof we refer to an identical technique used in [MECL12, Lemma 4.11]

Proof of Proposition 5.7. The boundary condition in (14a) is in a pointwise sense, and is an immediate consequence of the definition of the sequential exit-times introduced in Definition 3.1. Namely, for any initial state $x \in A^c_k$ we have $\Theta^{A_{k:n}}_k(t, x) = t$, and in light of Lemma 3.2 for all $i \in \{k, \cdots, n\}$

$$\Theta^{A_{k:n}}_i(t, x) = \Theta^{A_{k+1:n}}_i(t, x), \qquad \forall (t, x) \in [0, T_k] \times \partial A_k \bigcup \{T_k\} \times \mathbb{R}^d.$$

This leads to $X^{t, x; \boldsymbol{u}}_{\tau^k_k} = x$ on the boundaries, and yields to the pointwise boundary condition (14a).

The boundary condition (14b) is in a discontinuous viscosity sense, and is a standard boundary condition in this said context. To prove the assertion, let $\big(t_m, x_m, V_k(t_m, x_m)\big) \to \big(t, x, V^*_k(t, x)\big)$ where $t_m < T_k$. Invoking the DPP in Theorem 4.5 once again and introducing $\theta := \tau^k_{k+1}$ in (9a), we arrive at

$$V_k(t_m, x_m) \leq \sup_{\boldsymbol{u} \in \mathcal{U}_t} \mathbb{E}\Big[V^*_{k+1}\big(\tau^k_k, X^{t_m, x_m; \boldsymbol{u}}_{\tau^k_k}\big) \ell_k\big(X^{t_m, x_m; \boldsymbol{u}}_{\tau^k_k}\big)\Big].$$

Note that one can replace a sequence of policies in the above inequalities to attain the supremum running over all policies. This sequence, of course, depends on the initial condition (t_m, x_m). Hence, let us denote it via two indices $(\boldsymbol{u}_{m,j})_{j \in \mathbb{N}}$. One can deduce that there exists a subsequence of $(\boldsymbol{u}_{m_j})_{j \in \mathbb{N}}$ such that

$$V^*_k(t, x) = \lim_{m \to \infty} V_k(t_m, x_m) \leq \lim_{m \to \infty} \lim_{j \to \infty} \mathbb{E}\Big[V^*_{k+1}\big(\tau^k_k, X^{t_m, x_m; \boldsymbol{u}_{m,j}}_{\tau^k_k}\big) \ell_k\big(X^{t_m, x_m; \boldsymbol{u}_{m,j}}_{\tau^k_k}\big)\Big]$$

$$\leq \lim_{j \to \infty} \mathbb{E}\Big[V^*_{k+1}\big(\tau^k_k, X^{t_j, x_j; \boldsymbol{u}_{m_j}}_{\tau^k_k}\big) \ell_k\big(X^{t_j, x_j; \boldsymbol{u}_{m_j}}_{\tau^k_k}\big)\Big]$$

$$(24) \qquad \leq \mathbb{E}\Big[\lim_{j\to\infty} V_{k+1}^*\big(\tau_k^k, X_{\tau_k^k}^{t_j,x_j;\boldsymbol{u}_{m_j}}\big) \ell_k^*\big(X_{\tau_k^k}^{t_j,x_j;\boldsymbol{u}_{m_j}}\big)\Big]$$

$$(25) \qquad = V_{k+1}^*(t,x)\ell_k^*(x)$$

where (24) and (25) follow, respectively, from Fatou's lemma and the uniform continuity assertion in Lemma B.1. Let us recall that by Lemma B.1 we know $\tau_k^k(t_m,x_m) \to \tau_k^k(t,x) = t$ as $m \to \infty$ uniformly with respect to the policies $(\boldsymbol{u}_{m_j})_{j\in\mathbb{N}}$. It should be mentioned that the other side of discontinuous viscosity boundary condition (14b), i.e. $V_{k*}(t,x) \geq V_{k+1*}(t,x)\ell_{k*}(x)$, is an immediate result of pointwise boundary condition since the value functions $(V_k)_{k=1}^n$ and $(\ell_k)_{k=1}^n$ are all lower semicontinuous. $\qquad\square$

References

[AD90]   J.P. Aubin and G. Da Prato, *Stochastic viability and invariance*, Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie IV **17** (1990), no. 4, 595–613.

[AGA09]   Steven Andrews, Tuan Ginh, and Adam Arkin, *Stochastic models of biological processes*, Meyers RA, ed. Encyclopedia of Complexity and System Science (2009), 87308749.

[AP98]   J.P. Aubin and G. Da Prato, *The viability theorem for stochastic differential inclusions*, Stochastic Analysis and Applications **16** (1998), no. 1, 1–15.

[Aub91]   J.P. Aubin, *Viability Theory*, Systems & Control: Foundations & Applications, Birkhäuser Boston Inc., Boston, MA, 1991.

[BHKH05] Christel Baier, Holger Hermanns, Joost-Pieter Katoen, and Boudewijn R. Haverkort, *Efficient computation of time-bounded reachability probabilities in uniform continuous-time markov decision processes*, Theor. Comput. Sci. **345** (2005), no. 1, 2–26.

[Bor05]   V. S. Borkar, *Controlled diffusion processes*, Probability Surveys **2** (2005), 213–244 (electronic).

[BSS01]   A. Becskei, B. Seraphin, and L. Serrano, *Positive feedback in eukaryotic gene networks: cell differentiation by graded to binary response conversion*, EMBO J **20** (2001), no. 10, 2528–2535.

[BT11]   B. Bouchard and N. Touzi, *Weak dynamic programming principle for viscosity solutions*, SIAM J. Control Optim. **49** (2011), no. 3, 948–962.

[CCL11]   Debasish Chatterjee, Eugenio Cinquemani, and John Lygeros, *Maximizing the probability of attaining a target prior to extinction*, Nonlinear Analysis: Hybrid Systems (2011), http://dx.doi.org/10.1016/j.nahs.2010.12.003.

[Che00]   J. Cherry, *How to make a Biological Switch*, Journal of Theoretical Biology **203** (2000), no. 2, 117–133.

[CIL92]   M. G. Crandall, H. Ishii, and P. L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, American Mathematical Society **27** (1992), 1–67.

[CS98]   Y. Chitour and H. J. Sussmann, *Line-integral estimates and motion planning using the continuation method*, Essays on Mathematical Robotics, proceedings of the February 1993 workshop on Robotics held at the Institute for Mathematics and its Applications (IMA) at the University of Minnesota, Springer-Verlag, New York, 1998, pp. 91–126.

[EK86]   S.N. Ethier and T.G. Kurtz, *Markov processes: Characterization and convergence*, Wiley Series in Probability and Mathematical Statistics, John Wiley & Sons, Ltd., New York, 1986.

[ESKPG05] Hana El Samad, Mustafa Khammash, Linda Petzold, and Dan Gillespie, *Stochastic modelling of gene regulatory networks*, Internat. J. Robust Nonlinear Control **15** (2005), no. 15, 691–711.

[FS06]   W.H. Fleming and H.M. Soner, *Controlled Markov Processes and Viscosity Solution*, 3 ed., Springer-Verlag, 2006.

[GCC00]   T. S. Gardner, C. R. Cantor, and J. J. Collins, *Construction of a genetic toggle switch in Escherichia coli*, Nature **403** (2000), no. 6767, 339–42.

[Kal97]   O. Kallenberg, *Foundations of Modern Probability*, Probability and its Applications (New York), Springer-Verlag, New York, 1997.

[Kha]   Mustafa Khammash, *Modeling and analysis of stochastic biochemical networks*, Control Theory and System Biology.

[Kry09]   N.V. Krylov, *Controlled Diffusion Processes*, Stochastic Modelling and Applied Probability, vol. 14, Springer-Verlag, Berlin Heidelberg, 2009, Reprint of the 1980 Edition.

[KS91]   I. Karatzas and S.E. Shreve, *Brownian Motion and Stochastic Calculus*, 2 ed., Graduate Texts in Mathematics, vol. 113, Springer-Verlag, New York, 1991.

[MECL11] Peyman Mohajerin Esfahani, Debasish Chatterjee, and John Lygeros, *On a problem of stochastic reach-avoid set characterization for diffusions*, IEEE Conference on Decision and Control, dec. 2011, pp. 7069 –7074.

[MECL12]   _____, *On A Stochastic Reach-Avoid Problem and Set Characterization*, http://arxiv.org/abs/1202.4375, February 2012.

[Mit05]   I. Mitchell, *A toolbox of hamilton-jacobi solvers for analysis of nondeterministic continuous and hybrid systems*, Hybrid systems: computation and control (M. Morari and L. Thiele, eds.), Lecture Notes in Comput. Sci., no. 3414, Springer-Verlag, 2005, pp. 480–494.

[MSSO$^+$11] A. Milias, S. Summers, J. Stewart-Ornstein, I. Zuleta, D. Pincus, H. El-Samad, M. Khammash, and J. Lygeros, *In silico feedback for in vivo regulation of a gene expression circuit*, Nature Biotechnology **29** (2011), no. 12, 11141116.

[Øks03]   Bernt Øksendal, *Stochastic differential equations*, sixth ed., Universitext, Springer-Verlag, Berlin, 2003, An introduction with applications. MR 2001996 (2004e:60102)

[Pro05]   Philip E. Protter, *Stochastic integration and differential equations*, Stochastic Modelling and Applied Probability, vol. 21, Springer-Verlag, Berlin, 2005, Second edition. Version 2.1, Corrected third printing.

[RB98]   Richard Bass, *Diffusions and elliptic operators*, Probability and its Applications (New York), Springer-Verlag, New York, 1998.

[Ren99]   P. J. Reny, *On the existence of pure and mixed strategy nash equilibria in discontinuous games*, Econometrica **67** (1999), 1029–1056.

[SL10]   Sean Summers and John Lygeros, *Verification of discrete time stochastic hybrid systems: a stochastic reach-avoid decision problem*, Automatica J. IFAC **46** (2010), no. 12, 1951–1961. MR 2878218

[Sus91]   H. J. Sussmann, *Two new methods for motion planning for controllable systems without drift*, Proceedings of the First European Control Conference (ECC91) (Grenoble, France), 1991, pp. 1501–1506.

P. MOHAJERIN ESFAHANI AND JOHN LYGEROS ARE WITH THE AUTOMATIC CONTROL LABORATORY, ETH ZÜRICH, ETL I22, 8092 ZÜRICH, SWITZERLAND

E-mail address: {mohajerin,lygeros}@control.ee.ethz.ch

URL: http://control.ee.ethz.ch

D. CHATTERJEE IS WITH THE SYSTEMS & CONTROL ENGINEERING, INDIAN INSTITUTE OF TECHNOLOGY, BOMBAY, POWAI, MUMBAI 400076, INDIA

E-mail address: chatterjee@sc.iitb.ac.in

URL: http://www.sc.iitb.ac.in/~chatterjee